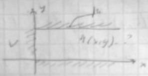
IV - 95 [illegible]



$$\begin{cases} \Delta u(x,y) = 0 \\ u(0,y) = V, y \in (0, b) \\ u(x,0) = 0, \quad x \in (0, +\infty) \\ u(x,b) = 0 \end{cases}$$

Решение

$$u = X(x)\, Y(y)$$

$$\frac{\partial^2 X}{\partial x^2} Y(y) + \frac{\partial^2 Y}{\partial y^2} \cdot X(x) = 0$$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda$$

$$\begin{cases} X''(x) - \lambda X(x) = 0 \\ Y''(y) - \lambda Y(y) = 0 \end{cases}$$

$$Y(0) = 0, \quad Y(b) = 0$$

$$\lambda = \left(\frac{\pi n}{b}\right)^2, \quad n = 1, 2$$

$$Y_n(y) = \sin \frac{\pi n}{b} y$$

$$X''(x) = \left(\frac{\pi n}{b}\right)^2 X(x)$$

$$X_n(x) = A e^{-\frac{\pi n}{b} x} + \cancel{B e^{\frac{\pi n}{b} x}}, \quad x \in (0, +\infty)$$

$$X_n(x) = A e^{-\frac{\pi n}{b} x}$$

[illegible]

$$X_n(x) Y_n(y) = A_n e^{-\frac{\pi n}{b} x} \sin\left(\frac{\pi n}{b} y\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} A_n e^{-\frac{\pi n}{b} x} \sin\left(\frac{\pi n}{b} y\right)$$

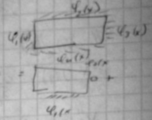
$$u(x,0) = 0, \quad u(x,b) = 0$$

$$\Delta u(x,y) = 0$$

$$u(0,y) = \sum_{n=1}^{+\infty} A_n \sin\left(\frac{\pi n}{b} y\right) = V$$

$$A_n = \frac{V \int_0^b dy \sin\left(\frac{\pi n}{b} y\right)}{b/2} = \frac{2V}{b} \frac{b}{\pi n} \cos \frac{\pi n}{b} y \Big|_b^0 =$$

$$= \frac{2V}{\pi n}\left(1 - (-1)^n\right)$$



$$u(x,y) = \frac{4V}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{\pi n}{b} x} \sin\left(\frac{\pi n}{b} y\right)$$

IV 113. $u\big|_{z=l} = 0$

$u\big|_{\rho=\rho_0} = 0$

$\frac{\partial u}{\partial z}\Big|_{z=0} = -q/æ$

$-æ \nabla u = \vec{q}$

$u_t = a^2 \Delta u$ – стационарн.

$\Delta u = 0$ – потом будет нелинейное стационарн.

$u = R(\rho) V(z)$

$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right)^{=-\lambda} \qquad \frac{V''(z)}{V(z)} = 0$$

$$\begin{cases} \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + \lambda R(\rho) = 0 \\ R(\rho_0) = 0 \end{cases} \qquad \Big\{ V''(z) = \lambda V(z)$$

$$\rho^2 \frac{d^2}{d\rho^2} R(\rho) + \rho \frac{d}{d\rho} R(\rho) + \lambda \rho^2 R(\rho) = 0$$

$$R(\rho) = A J_0(\sqrt{\lambda}\rho) + B N_0(\sqrt{\lambda}\rho)$$

$$|R(0)| < \infty \Rightarrow B = 0$$

$$R(\rho_0) = A J_0(\sqrt{\lambda}\rho_0) = 0 \Rightarrow \sqrt{\lambda}\rho_0 = \mu_n^{(0)},\ n = 1, 2 \ldots$$

$$\lambda_n = \left(\frac{\mu_n^{(0)}}{\rho_0}\right)^2,\ R_n(\rho) = J_0\left(\frac{\mu_n^{(0)}}{\rho_0}\rho\right)$$

$$V_n(z) = A_n e^{\frac{\mu_n^{(0)}}{\rho_0} z} + B_n e^{-\frac{\mu_n^{(0)}}{\rho_0} z}$$

$$u(\rho, z) = \sum_{n=1}^{+\infty}\left(A_n e^{\frac{\mu_n}{\rho_0} z} + B_n e^{-\frac{\mu_n}{\rho_0} z}\right) J_0\left(\frac{\mu_n}{\rho_0}\rho\right)$$

$$u(\rho, l) = 0 \qquad 0 \le \rho \le \rho_0 \qquad = C_n \operatorname{sh}\left(\frac{\mu_n}{\rho_0}(z - l)\right) + \cancel{D_n \operatorname{ch}\left(\frac{\mu_n}{\rho_0}(z - l)\right)}$$

$$u(\rho, z) = \sum_{n=1}^{+\infty} C_n \operatorname{sh}\left(\frac{\mu_n}{\rho_0}(z - l)\right) J_0\left(\frac{\mu_n}{\rho_0}\rho\right) \quad \begin{cases} J_0(\mu_n) = 0, \\ \mu_n > 0 \end{cases}$$

$$-\frac{q}{\kappa} = \sum C_n \frac{\mu_n}{\rho_0} \operatorname{ch}\left(\frac{\mu_n}{\rho_0} l\right) J_0\left(\frac{\mu_n}{\rho_0}\rho\right)$$

$$C_n \frac{\mu_n}{\rho_0} \operatorname{ch}\left(\frac{\mu_n}{\rho_0} l\right) = -\frac{q}{\kappa} \frac{\int_0^{\rho_0} d\rho\, \rho\, J_0\left(\frac{\mu_n}{\rho_0}\rho\right)}{\left\| J_0\left(\frac{\mu_n}{\rho_0}\rho\right)\right\|^2} = -\frac{q}{\kappa} \frac{\rho_0^2}{\mu_n^2}$$

$$\int_0^{\mu_n} dx\, x J_0(x) \Big/ \frac{\rho_0^2}{2} J_1^2(\mu_n) = -\frac{2q}{\kappa \mu_n} \frac{\mu_n J_1(\mu_n)}{J_1^2(\mu_n)}$$

$$C_n = \frac{-2q}{\kappa}\rho_0 \frac{1}{\mu_n^2 \operatorname{ch}\left(\frac{\mu_n}{\rho_0} l\right) J_1(\mu_n)}$$

$$\cancel{u(\rho, z) =}$$

$$u = -\frac{2q\rho_0}{\kappa} \sum_{n=1} \frac{\operatorname{sh}\left[\frac{\mu_n^{(0)}}{\rho_0}(z - l)\right] J_0\left(\frac{\mu_n^{(0)}}{\rho_0}\rho\right)}{\operatorname{ch}\left[\frac{\mu_n^{(0)}}{\rho_0} l\right]\left[\mu_n^{(0)}\right]^2 J_1(\mu_n^{(0)})}$$

$$= \frac{1}{2\pi}\int_{-\pi} d\varphi\, e^{i(x\sin\varphi - p\varphi)} = f_p(x)$$

$p\in\mathbb{Z}$

Примеры

№ 712 $\quad u_{tt} = a^2 u_{xx} - 2h u_t - b^2 u \quad (h>0),\ x\in(0,l),\ t>0$

$u = u(x,t),\ u(0,t) = A\ ;\ u(l,t) = 0 \quad$ (гр. у.)

$u(x,0) = 0\ ,\ u_t(x,0) = 0$

Реш. $\quad u(x,t) = f(x) + v(x,t)$

$$\begin{cases} u(0,t) = f(0) + v(0,t) = A \\ u(l,t) = f(l) + v(l,t) = 0 \end{cases} \qquad v(0,t) = 0,\quad v(l,t) = 0$$

$$\begin{cases} u(x,0) = f(x) + v(x,0) \\ v_t(x,0) = 0 \end{cases}$$

$$a^2 f''(x) - b^2 f(x) = 0$$

$f(0) = A$
$f(l) = 0$

$$f(x) = A\,\frac{\operatorname{sh}\frac{b}{a}(l-x)}{\operatorname{sh}\left(\frac{b}{a}l\right)}$$

$$a^2 f_{xx} = b^2 f$$

$$f''(x) = \frac{b^2}{a^2} f(x)$$

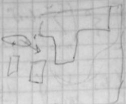
$v_{tt} - a^2 v_{xx} - 2h v_t - b^2 v,\ x \in (0,l),\ t>0$

гр.у $v(0,t)=0,\ v(l,t)=0$

$v_t(x,0)=0;$

$v_t(x,0)=0$

$v(x,0)=f(x)$



$$f(x)=\frac{A}{sh(\frac{c}{a}l)}\, sh\frac{c}{a}(l-x)$$

$\cancel{v_{tt}=a^2v_{xx}}$

$V=T(t)P(x)\ /\ \cancel{T(t)P(x)}$

$P(0)=0=P(l)\qquad P(0)=P(l)=0$

$$P(x)\,T''(t)=a^2\,T(t)\,P''(x)-2h\,P(x)\,T'(t)-b^2\,T(t)\,P(x)$$

$$\frac{T''(t)}{a^2T(t)}+\frac{2h}{a^2}\frac{T'(t)}{T(t)}+\frac{b^2}{a^2}=\frac{P''(x)}{P(x)}=-\lambda$$

$$\begin{cases} P''(x)+\lambda P(x)=0;\ x\in(0,l) \\ P(0)=0,\ P(l)=0 \end{cases}$$

$$P_n(x)=\sin\left(\pi n\frac{x}{l}\right)\qquad \lambda_n=\left(\frac{\pi n}{l}\right)^2$$

$n=1,2$

$$T''(t)+2h\,T'(t)+(b^2+a^2\lambda_n)\,T(t)=0$$

$T=e^{kt}$

$$k^2+2kh+(b^2+a^2\lambda_n)=0$$

$$k_{1,2}=-h\pm\sqrt{h^2-}$$

$$\eta = C_n e^{(-h+\sqrt{h^2-k^2\lambda_n})t} + D_n e^{(-h-\sqrt{h^2-k^2\lambda_n})t} =$$

$$= e^{-ht}\left[E_n ch\sqrt{h^2-b^2-a^2\lambda_n}\,t + F_n sh\sqrt{h^2-b^2-a^2\lambda_n}\,t\right]$$

$$V = \sum_{n=1}^{\infty} e^{-ht}\left[E_n ch\sqrt{h^2-b^2-a^2\left(\frac{\pi n}{l}\right)^2}t + F_n sh\sqrt{h^2-b^2-a^2\left(\frac{\pi n}{l}\right)^2}t\right] \qquad \lambda_n = \left(\frac{\pi n}{l}\right)^2$$

$$\sin\left(\pi n \frac{x}{l}\right);$$

1) $$\sum_{n=1}^{\infty} E_n \sin\frac{\pi n}{l}x = \cancel{f(x)} - A\frac{sh\frac{b}{a}(l-a)}{sh\frac{b}{a}l}$$

$$E_n = -\frac{2}{l}A\int_0^l A\,\frac{sh\frac{b}{a}(l-a)}{sh\frac{b}{a}l}\sin\frac{\pi n}{l}x\,dx = I$$

2) $$\sum_{n=1}^{\infty}\left[F_n - hE_n\right]\sin\frac{\pi n}{l}x$$

$$F_n\sqrt{h^2-b^2-a^2\left(\frac{\pi n}{l}\right)^2} = hE_n$$

$$F_n = E_n\frac{1}{\sqrt{h^2-b^2-a^2\left(\frac{\pi n}{l}\right)^2}}$$

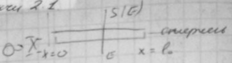
$$I = -\frac{l}{\pi n}\,sh\frac{b}{a}(l-a)\cos\frac{\pi n}{l}a\Big|_0^l + \frac{l}{\pi n}\int_0^l \cos\frac{\pi n}{l}x\cdot ch\frac{b}{a}(l-a)\cdot\frac{b}{a}\cdot dx$$

$$= \frac{l}{\pi n}sh\frac{b}{a}l - \left(\frac{l}{\pi n}\right)^2\sin\frac{\pi n}{l}x\Big|_0^l - \left(\frac{l}{\pi n}\right)^2\left(\frac{b}{a}\right)^2 I$$

$$I = \frac{\frac{l}{\pi n}\,sh\frac{b}{a}l}{1+\left(\frac{lb}{\pi n a}\right)^2} \qquad E_n = -\frac{2A}{sh\frac{b}{a}l}\cdot\frac{\frac{l}{\pi n}sh\frac{b}{a}l}{1+\left(\frac{lb}{\pi n a}\right)^2} =$$

$$= -\frac{2A}{\pi n}\frac{1}{1-\left(\frac{lb}{\pi n a}\right)^2}; \qquad u(x,t) =$$

Задачи 2.1



$\rho = \rho(\vec{r}, t)$
$S = S(\vec{r}, t)$
$\vec{v} = \vec{v}(\vec{r}, t)$

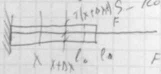
Эйлеровы координ.

Лагранжевы координ. – поле жидкого мал-частиц

$u = u(x,t)$ – смещение точки с координатой x в момент времени t

$X(x,t) = x + u(x,t)$

$v(x,t) = \frac{\partial}{\partial t} X(x,t)$



$F = S \cdot E \frac{l - l_0}{l} = SE \cdot \mathcal{E}$ –

Закон Гука

$X(x+\Delta x, t) = x + \Delta x + u(x + \Delta x, t)$

$X(x,t) = x + u(x,t)$

$x + \Delta x + u(x + \Delta t, t) - x - u(x,t) \approx \Delta x$

$$\varepsilon(x,t) = \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$

$$\varepsilon(x,t) = \frac{u_x(x,t) - u(x,t)}{\Delta x}$$

$$T(x,t) = \frac{E}{S} \cancel{\varepsilon(x,t)}\, u_x(x,t)$$

$\Delta m \cdot u_{tt}(x,t) = T(x+\Delta x,t) - T(x,t)$ — теор

$$\Delta m\, u_{tt}(x,t) = SE\left(u_x(x+\Delta x,t) - u_x(x,t)\right)$$

$$\rho S \Delta x\, u_{tt}(x,t) = SE \quad \Delta x \to 0$$

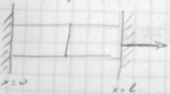
$$u_{tt}(x,t) = \left(\frac{\rho}{E}\right)^{-1} u_{xx}(x,t), \quad \left(\frac{\rho}{E}\right)^{-1} = a^2 \qquad a = \sqrt{\frac{E}{\rho}}$$

, одномірна задача

Крайові задачі

$$u_{tt}(x,t) = \frac{E}{\rho} u_{xx}(x,t) = a^2 u_{xx}(x,t)$$

а) коли кінці закріплені жорстко



не має деформації $\rho = \rho_0$

1) Запишемо граничні умови $\quad x \in (0,l)$

$\cancel{u(0)=0}$ $\quad u(0,t)=0$

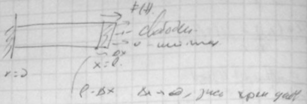
$u(l,t)=0 \qquad$ 1-а крайова задача

$u(0,t) = \mu_1(t)$

$u(l,t) = \mu_2(t)$

б) концы свободны

Не имеет внешних сил



Задачи

Подставим →

$$\Delta m\, u_{tt}(l,t) = f(t) - SE\, u_x(l-\Delta x, t)$$

$$\Delta m = \rho S \Delta x$$

$$\rho S \Delta x\, u_{tt}(l,t) = f(t) - SE\, u_x(l-\Delta x, t)$$

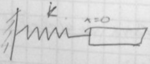
$$f(t) = SE\, u_x(l,t)$$

$$u_x(l,t) = \frac{1}{SE} f(t) \qquad m \to 0$$

$$\rho(x)\, u_{tt} = T_0 u_{xx} - V u_t(x,t) \cancel{+f}$$

$$u_x(l,t) = 0$$

в) закреплен упруго

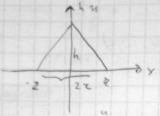


$$F = -kx$$

27.09.11

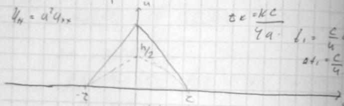
Метод характеристик

$\boxed{\text{I}}$. -52.

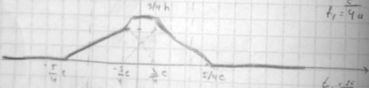


$u_{tt} = a^2 u_{xx}$

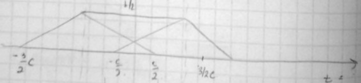
$t_k = \frac{kc}{4a}$ $\quad t_1 = \frac{c}{4a}$ $\quad \Delta t_1 = \frac{c}{4}$



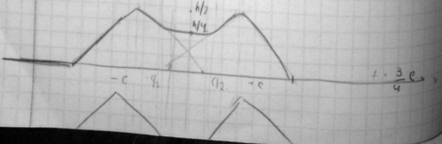
$t_1 = \frac{c}{4a}$



$t_2 = \frac{2c}{4a}$



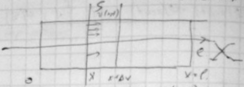
$t =$



$t = \frac{3}{4} c$

2.4.

За Δt протекает опр. кол-во газа



$\rho = \rho(x,t)$

$p = p(x,t)$

Нет диссипации

$v = v(x,t)$ – производная от смещ.

$v(x,t) = u_t(x,t)$

$\rho=\rho_0,\ p=p_0$ – Равновесие жидкости

ρ_0 – смещен.

p_0 – смещен.

$v = 0$

Выводим уравнение движения

$v = v(x,t)$

1. Уравнение непрерывности (сохранение массы)

$$\Delta m = \rho(x,t)\cdot v(x,t)\cdot S\Delta t - \rho(x+\Delta x,t)\,v(x+\Delta x,t)\,S\Delta t =$$

$$m = \rho V = \rho\cdot S\Delta x = \Delta\rho\, S\Delta x = S\Delta x\left[\rho(x,t+\Delta t) - \rho(x,t)\right]$$

$/\ \Delta x\Delta t\cdot S$

$$-\frac{\rho(x+\Delta x,t)v(x+\Delta x,t) - \rho(x,t)v(x,t)}{\Delta x} = \frac{\rho(x,t+\Delta t)-\rho(x,t)}{\Delta t}$$

$\Delta x\to 0,\ \Delta t\to 0$

$$\boxed{\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0}$$ Уравнение непрерывности

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\vec{v}) = 0$$

$\rho=\rho_0,\ p=p_0$ -

② Уравнение Ньютона (2-й закон) $\bar{v}(\bar{r}, t)$

— кусок движется со скоростью $\bar{v}$

$$\Delta m\, a = F$$

$$\Delta m = \rho S \Delta x$$

$$a \neq \frac{d\bar{v}}{dt}$$

ускорение жидкости, который движется с жидкостью

За Δt он переместится на $\Delta \bar{r} = \bar{v}\,\Delta t$

$$\bar{a}(\bar{r}, t) = \lim_{\Delta t \to 0} \frac{\Delta \bar{v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\bar{v}(\bar{r} + \bar{v}\Delta t, t + \Delta t) - \bar{v}(\bar{r}, t)}{\Delta t} =$$

$$= \lim_{\Delta t \to 0} \left[\bar{v}(\bar{r}, t) + \frac{\partial}{\partial \bar{r}} \Delta t\, \bar{v}(\bar{r}, t) + \frac{\partial v}{\partial t} \Delta t - \bar{v}(\bar{r}, t) \right] =$$

$$\left[v(x + \Delta x, y + \Delta y) = v(x, y) + \Delta x \frac{\partial}{\partial x} v + \Delta y \frac{\partial v}{\partial y} + \dots \right]$$

$$= \lim_{\Delta t \to 0} \left[\frac{\partial}{\partial r} \Delta t\, \bar{v}(\bar{r}, t) + \frac{\partial \bar{v}}{\partial t} \Delta t \right] \frac{1}{\Delta t}$$

$$\bar{a}(\bar{r}, t) = \frac{\partial \bar{v}(\bar{r}, t)}{\partial t} + (\bar{v} \cdot \nabla)\, \bar{v}(\bar{r}, t)$$

$$\boxed{a(x, t) = \frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x}}$$

Массив сил.

$$a = \frac{dv}{dt}$$

$$\rho S \Delta x = \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) = \underset{\text{- давление}}{P(x, t) \cdot S} - P(x + \Delta x, t) S$$

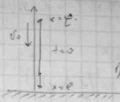
Из-за разницы давлений возникает сила.

$$\boxed{\rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} \right) = -\frac{\partial P}{\partial x}}$$

II. 127.

$0 \le x \le \ell$

v_0



$u = u(x,t),\ x \in (0,\ell)$.

1) $u(0,t)=0$;

2) $u_x(\ell,t)=0$;

На край концы никаких сил не действуют

$u_{tt}(x,t) = a^2 u_{xx}(x,t)$

$x \in (0,\ell),\ t>0$.

НУ

$$\begin{cases} u(x,0)=0 \\ u_t(x,0) = -v_0. \end{cases}$$

$u(x,t) = V(x)T(t)$

$u_{tt} = T''V$

$u_{xx} = V''T$

$T''(t)V_x = a^2 T(t) V''(x)$

$$\frac{T''(t)}{a^2 T(t)} = \frac{V''(x)}{V(x)} = -\lambda = const$$

$$\begin{cases} V''(x) + \lambda V(x) = 0. \\ T''(t) + \lambda a^2 T(t) = 0. \end{cases}$$

1) $u(0,t)=0$

$V(0)=0$.

2) $u_x(\ell,t)=0$

$V'(\ell)\,T(t)=0$.

$V'(\ell)=0$.

$V(x) = A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x$

$0 = A\cdot 0 + B \Rightarrow B=0$

$0 = \sqrt{\lambda}A\cos\sqrt{\lambda}\ell - \sqrt{\lambda}B\sin\sqrt{\lambda}\ell.$

$$\cos\sqrt{\lambda}\,l = 0 \Rightarrow \Rightarrow \sqrt{\lambda_n}\,l = \frac{\pi}{2} + \pi n,\ n = 0, 1, 2 \ldots$$

$$\lambda_n = \left(\frac{\pi}{2l}(2n+1)\right)^2$$

$$X_n(x) = A_n \sin\left(\frac{\pi}{2l}(2n+1)x\right)$$

$$T_n(t) = C_n \sin\left(\frac{\pi(2n+1)a}{2l}t\right) + D_n \cos\left(\frac{\pi(2n+1)a}{2l}t\right)$$

$$u_n(x,t) = \sin\left(\frac{\pi(2n+1)}{2l}x\right)\left\{C_n \sin\left(\frac{\pi(2n+1)a}{2l}t\right) + D_n \cos\left(\frac{\pi(2n+1)a}{2l}t\right)\right\}$$

$$u(x,t) = \sum_{n=0}^{\infty} \sin\frac{\pi(n+1/2)x}{l}\left\{C_n \sin\left(\frac{\pi(n+1/2)a}{l}t\right) + D_n \cos\left(\frac{\pi(n+1/2)a}{l}t\right)\right\}$$

1) $u(x,0) = 0$

$$0 = \sum \sin\left(\frac{\pi(n+1/2)}{l}x\right) D_n$$

$\forall n\ D_n = 0$ - удобн.

2) $$-v_0 = \sum_{n=0}^{\infty} \sin\left(\frac{\pi(n+1/2)}{l}\right) C_n \frac{\pi(n+1/2)}{l} a \cos\left(\frac{\pi(n+1/2)}{l}\right)$$

$$-v_0 \left| \sin\left(\frac{\pi(n+1/2)}{l}x\right)\right.$$

$$\sum_{n=0}^{\infty} C_n \frac{\pi(n+1/2)a}{l} \int_0^l dx \left\{\sin\left(\frac{\pi(n+1/2)}{l}x\right\} \sin\frac{\pi(m+1/2)}{l}x\right\}$$

$$= -\int_0^l dx\, v_0 \sin \frac{\pi(m+1/2)}{l} x$$

$$I = \frac{1}{2}\int_0^l dx \left\{ \cos \frac{\pi(n-m)}{2} x - \cos \frac{\pi(n+m+1)}{l} x \right\} =$$

$$= \frac{1}{2}\frac{l}{\pi(n-m)} \sin\left(\frac{\pi(n-m)}{l}\right)^2 \Big|_0^l - \frac{1}{2}\frac{l}{\pi(m+n+1)} \sin\left(\frac{\pi(n+m+1)}{l}\right)\Big|_0^l -$$

$$-\frac{1}{2}\frac{l}{\pi(m+n+1)} \sin\left(\frac{\pi(n+m+1)}{l}\right)\Big|_0^l = 0+0=0 \Big|^{m=n} \quad I = \frac{1}{2}\int_0^l dx \left(1 - \cos\frac{\pi(2n+1)}{l} x\right) =$$

$$= \frac{l}{2} - \frac{l}{2\pi(2n+1)} \sin \frac{\pi(2m+1)}{2}\Big|_0^l = \frac{l}{2}$$

$$u(x,t) = \sum_{n=0}^{\infty} \sin\left(\frac{\pi(n+1/2)}{l} x\right) \left(\frac{C_n \sin(\pi(n+1/2)a}{l} t\right) -$$

$$-\int_0^l dx\, v_0 \sin \frac{\pi(m+1/2)}{l} x = C_m \frac{1}{2}\pi\left(m+\frac{1}{2}\right)a.$$

$$\tilde{I} = v_0 \frac{l}{\pi(m+\frac{1}{2})} \cos \frac{\pi(m+\frac{1}{2})}{l} x \Big|_0^l = -v_0 \frac{l}{\pi(m+\frac{1}{2})}$$

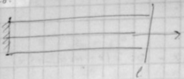
$$u(x,t) = \frac{2 v_0 l}{\pi^2 a} \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})} \sin\left(\frac{\pi(n+\frac{1}{2})}{l} x\right) \left(\frac{\pi(n-1/2)}{l} t\right).$$

$$-v_0 \frac{l}{\pi(m+\frac{1}{2})} = \frac{1}{2} C_m \pi\left(m-\frac{1}{2}\right)a. \qquad T = u_t(x,t)$$

$$C_n = -v_0 \frac{2l}{\pi^2(m+\frac{1}{2})a}. \qquad T(x,t) = x\, u(x,t)$$

$$\tau(t)$$

2.128.



$(t>0)$

$\Delta m\, u_{tt}(l,t) = F_0 - T(l-\Delta x, t) =$

$= F_0 - ES\, u_x(l-\Delta x, t)$

$\Delta m = \rho S \Delta x \quad \Delta x \to 0.$

$u_x(l,t) = \dfrac{F_0}{ES}$

$u(x,0) = 0 = u_t(x,0) \quad u(0,t) = 0$

$u_{tt}(x,t) = a^2 u_{xx}(x,t).$

$\tilde{u}(x,t) = X(x)\,T(t).$

$\dfrac{X''(x)}{X(x)} = \dfrac{T''(t)}{a^2 T(t)} = -\lambda = const.$

$$\begin{cases} X''(x) + \lambda X(x) = 0. \\ T''(t) + \lambda a^2 T(t) = 0. \end{cases}$$

$t > 0$

$u(x,t) = v(x,t) + h(x)$

$v(x,t) = X(x)\,T(t)$

$h(x) = ax + b.$

$h'(l) = \dfrac{F_0}{ES} = a.$

$h(0) = 0 = b$

$h''(x) = 0$

$v_{tt}(x,t) = a^2 \cdot b \cos(x,t)$

$u_x(l,t) = v_x(x,t)$

$u_x(l,t) = v_x(l,t) + \frac{F_0}{ES} = \frac{F_0}{ES}$

$v_x(l,t) = 0.$

$u(x,0) = v(x,0) + \frac{F_0}{ES} x = 0.$

$u(0,t) = v(0,t) = 0$

$v(x,0) = \frac{-F_0}{ES} x = 0$

$v(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(n + \frac{1}{2}\right)\pi \frac{at}{l} \right) + B_n \sin\left(n + \frac{1}{2}\right)\pi \frac{at}{l}$

$\sin\left(n + \frac{1}{2}\right)\pi \frac{x}{l}$

18.10.2011

90

$u_{tt} = a^2 u_{xx} + b\,x(x-l)$, $x \in (0,l)$, $t > 0$

$u(x,0) = 0$ $\quad u_t(x,0) = 0$, $x \in (0,l)$.

~~$u_t(x,0)$~~ $= 0$ $\quad u(l,t) = 0$, $t > 0$

$u(0,t) = 0$

$u(x,t) = v(x,t) + w(x)$

Хотим, чтобы $w(x)$: $\begin{cases} w_{tt} = a w_{xx} + b x(x-l) \\ w(0) = w(l) = 0 \end{cases}$

$W_{xx} = -\frac{b}{a^2} x(x-l)$

$w(x) = \alpha x^4 + \beta x^3 + \gamma x^2 + \sigma x + \varepsilon$

$12\alpha x^2 + 6\beta x + 2\gamma = -\frac{b}{a^2} x^2 + \frac{b}{a^2} l x.$

$12\alpha = -\frac{b}{a^2} \Rightarrow \alpha = -\frac{b}{12 a^2}$

$6\beta = \frac{b}{a^2} l$ $\quad \gamma = 0$

ГАРМО...

ТАМ

$$W(v) = -\frac{b}{a}$$

$$W(x) = -\frac{b}{12a^2}x^4 + \frac{b^2}{6a^2}bx^3 + \delta x + \varepsilon$$

$$W(0) = \varepsilon = 0$$

$$W(l) = \frac{-bl^4}{12a^2} + \frac{bl^4}{6a^2} + \delta l = 0$$

$$W(x) = \frac{b}{12a}\left(-x^4 + 2lx^3 - l^3x\right)$$

$$v_{tt} + \cancel{W_{tt}}^{\,0} = a^2\left(v_{xx} + w_{xx}\right) + bx(x-l) = a^2 v_{xx} \Rightarrow \quad v_{tt} = a^2 v_{xx}$$

$$v(0,t) + \cancel{W(0)}^{\,0} = 0 \Rightarrow v(0,t) = 0$$

$$v(l,t) = 0$$

I-ого рода

$$v(x,0) + w(x) = 0, \quad v(x,0) = -w(x)$$

$$v_t(x,0) = 0.$$

$$v(x,t) = \sum_{n=1}^{\infty}\left(A_n \cos \omega_n t + \cancel{B_n}^{\,0} \sin \omega_n t\right)\sin\frac{\pi n}{l}x, \quad \omega_n = \frac{\pi n}{l}$$

Усл. нач. данн.

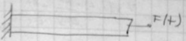
$$A_n = \frac{2}{l}\int_0^l dx\left(-w(x)\right)\sin\left(\frac{\pi n}{l}x\right)$$

$$A_n = -\frac{2}{l}\int_0^l dx\, w(x)\sin\left(\frac{\pi n}{l}x\right)$$

$= \frac{2}{l} w(x) \cos\left(\frac{\pi n}{l} x\right)$

147.

$F(t) = At \ (A = \mathrm{const})$



$u(x,0) = 0, \quad u_t(x,0) = 0$

$u(0,t) = 0 \qquad u_x(l,t) = \frac{F(t)}{ES} = \frac{At}{ES}$

$u_x(l,$

$u_{tt} = a^2 u_{xx}, \ x \in (0,l), \ t > 0$

Хотим

$$\begin{cases} w(0,t) = 0; \ w_x(l,t) = \frac{A}{ES} t \end{cases}$$

$w_{tt} = a^2 w_{xx}$

$w(x,t) = C(x)t + D(x)$

$0 = a^2(C''(x)t + D''(x))$

$$\begin{cases} C''(x) = 0 \\ D''(x) = 0 \end{cases} \qquad \begin{matrix} C(x) = Gx + H \\ D(x) = Jx + K \end{matrix}$$

$w(x,t) = Gxt + Ht + Jx + K$

$w(x,t) = Gxt + Ht + Jx + \mathcal{E}$

$w(0,t) = 0$

$w(0,t) = Ht + K = 0 \Rightarrow H, K = 0$

$w(l,t) = \frac{At}{ES} \qquad \frac{At}{ES} = Gt + J \quad \forall t >$

$t = 0, \ J = 0 \quad \frac{A}{ES} = G; \ w(x,t) = \frac{A}{ES}$

$v(x,0) + w(x,0) = 0$ $\qquad\qquad v_t(x,0) = -\frac{Av}{ES}$

$v_t(x,0) + w_t(x,0) = 0$

$v_{tt} = a^2 v_x$

$v(0,t) = 0.$ $\qquad\qquad v_x(l,t) = \frac{At}{ES} = \frac{At}{ES}$

$v_x(l,t) = 0$

$\cancel{v(x,t) = \sum_{n=0}^{+\infty} \left(A_n \cos \ldots \right.}$

$$v(x,t) = \sum_{n=0}^{+\infty} \left(A_n \cos \frac{\pi (n+\frac{1}{2})}{l} at + \right.$$

$$\left. + B_n \sin \frac{\pi (n+\frac{1}{2})}{l} at \right) \sin \frac{\pi (n+\frac{1}{2})}{l} x$$

II -146

N - 85 — смирнов

N - 91 —

II 134 $u_{tt} = a^2 u_{xx}$

$\varphi(r,t) = \int$

Почему кар

$\int_0 d\tau$

$\sin \lambda \sin \beta$

$= \frac{1}{4i^2} (e$

$= -\frac{1}{4} (e^{-}$

$= -\frac{1}{2} (e^{-2}$

$2 \int_0^1 d\tau$

2108.

$u_{tt} = a^2 u_{xx} + Ae^{-\alpha t} \sin \frac{\pi x}{l}$ $(\alpha > 0,\ A = const)$, $x \in (0, l), t > 0$

ГКУ $u(0,t) = 0$, $u(l,t) = 0$, $t > 0$

$V_n(x) = \sin \frac{\pi n}{l} x$

НУ $u(x,0) = 0$, $u_t(x,0) = 0$

$u(x,t)$ - ?

Найти такое решение

$u_2(x,t) = f(t) \sin \frac{\pi x}{l}$ $\quad u(x,t)$ - ?

$$f''(t) \sin \frac{\pi x}{l} = -a^2 f(t) \sin \frac{\pi}{l} x \cdot \frac{\pi^2}{l^2} + Ae^{-\alpha t} \sin \frac{\pi x}{l}$$

$$f''(t) + a^2 f(t) \frac{\pi^2}{l^2} = Ae^{-\alpha t}$$

$$f''(t) + \left(\frac{a\pi}{l}\right)^2 f(t) = 0$$

$$f_0(t) = B \cos\left(\frac{a\pi}{l} t\right) + C \sin\left(\frac{a\pi}{l} t\right)$$

$$f_* = De^{-\alpha t}$$

$$e^{-\alpha t} D\alpha^2 + \left(\frac{a\pi}{l}\right)^2 e^{-\alpha t} D = Ae^{-\alpha t}$$

$$D = \frac{A}{\alpha^2 + \left(\frac{\pi a}{l}\right)^2}$$

$$f = B \cos\left(\frac{\pi a}{l} t\right) + C \sin\left(\frac{\pi a}{l} t\right) + \frac{A}{\alpha^2 + \left(\frac{\pi a}{l}\right)^2} e^{-\alpha t}$$

$f(0) = 0$; $f'(0) = 0$

$0 = f(0) = B + \dfrac{A}{\lambda^2 + \left(\frac{\pi a}{l}\right)^2}$; $B = \dfrac{-A}{\lambda^2 + \left(\frac{\pi a}{l}\right)^2}$

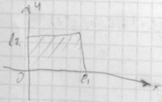
$f'(0) = C\dfrac{\pi a}{l} - \dfrac{A\lambda}{\lambda^2 + \left(\frac{\pi a}{l}\right)^2} = 0$; $C = \dfrac{A\lambda l}{\pi a} \Big/ \left(\lambda^2 + \left(\frac{\pi a}{l}\right)^2\right)$

Ответ: $f = \dfrac{-A}{\lambda^2 + \left(\frac{\pi a}{l}\right)^2} \cos\left(\frac{\pi a}{l}\right)^2 + \dfrac{A\lambda l}{\pi a} \Big/ \left(\lambda^2 + \left(\frac{\pi a}{l}\right)^2\right) \sin\left(\frac{\pi a}{l}\right)$

$+ \dfrac{A}{\lambda^2 + \left(\frac{\pi a}{l}\right)^2} e^{-\lambda t} \Big\}$

5.10.

6.47.



$u = u(x, y, t)$

$u|_{\Gamma} = 0$ (гр. условия)

$u(x, y, 0) = \varphi(x, y) - A\,xy(l_1 - x)(l_2 - y)$

$u_t(x, y, 0) = \psi(x, y) = 0.$

$u_{tt} = a^2(u_{xx} + u_{yy})$, $a^2 = \dfrac{T}{\rho}$

$u(x, y, t) \to ?$

$u_z(x, y, t) = T(t)\,V(x, y)$

$T''(t)\, v(x,y) = a^2 \left(T(t)\, v_{xx}(x,y) + T(t)\, v_{yy}(x,y) \right)$

$T_{tt} V = a^2 T (V_{xx} + V_{yy})$

$\frac{T_{tt}}{a^2 T} = \frac{V_{xx} + V_{yy}}{V} = -\lambda$

$u(x,0,t) = 0 \Rightarrow T(t) V(x,0) = 0$

$T(t) V(x,l) = 0$

$\begin{cases} (V_{xx} + V_{yy}) = -\lambda V \\ V|_{\Gamma} = 0 \end{cases} \quad V = f(y,x)$

$V(x,y) = X(x)\, Y(y)$

$X_{xx} Y + Y_{yy} X = -\lambda X Y$

$\frac{X_{xx}}{X} + \frac{Y_{yy}}{Y} = -\lambda$

$-\mu \quad -\eta$

$\begin{cases} X_{xx} + \mu X = 0 \\ Y_{yy} + \eta Y = 0 \end{cases}$

$T\, X \cdot Y(0) = 0$

$T X Y(l) = 0$

$T\, X(0) Y(0) = 0$

$T\, X(l_1) Y(0) = 0$

$X_n = \sin \frac{\pi n}{l_1} x, \quad \mu_n = \left(\frac{\pi n}{l_1} \right)^2$

$Y_m = \sin \frac{\pi m}{2} y, \quad \eta_m = \left(\frac{\pi m}{l} \right)^2$

$V(x,y) = \sin \pi$

$\to \quad \lambda_{nm} = \left(\frac{\pi n}{l_1}\right)^2 + \left(\frac{\pi m}{l_2}\right)^2$

$$u = \sum_{n=0}^{\infty} \left(A_{nm} \sin\sqrt{\lambda_{nm}} + B_{nm} \cos\sqrt{\lambda_{nm}}\, t\right) \sin\frac{\pi n}{l} x$$

$\sin\frac{\pi m}{l_2} y$

$$u(x,y) = \sum_{n,m=1}^{\infty} B_{nm} \sin\frac{\pi n}{l_1} x + \sin\frac{\pi m}{l_2} y =$$

$$= A\, xy(l_1 - x)(l_2 - y) \quad \Big| \quad \sin\frac{n\pi}{l}, \quad \sin\frac{\pi l}{l}$$

$$\sum_{n=0}^{\infty} B_n \int_0^{l}$$

$$I(l, p) = -\frac{l}{\pi p}\left(x(l-x)\cos\frac{\pi p}{l}x \Big|_0^l - \int \cos\frac{\pi p}{l} x\,(l - 2x)\,dx \right) =$$

$$= \left(\frac{l}{\pi p}\right)\left\{ \sin\frac{\pi p}{l} x(l - 2x)\Big|_0^l - \int (-2) \sin\frac{\pi p}{l} x\, dx \right\} =$$

$$= -2\left(\frac{l}{\pi p}\right)^2 \cos\frac{\pi p}{l}x \Big|_0^l = -2\left(\frac{l}{\pi p}\right)^3 \left[\cos\pi p - 1\right]$$

$$\sum_{n=1} c_{nm} \quad \frac{l_1 l_2}{n}\, \delta_{np}\, \delta_{mq} = A_1\, \frac{4/l_1 l_2}{\pi/4} \Big)^3 \left[\cos\pi p - 1\right]\left[\cos\pi q - 1\right]$$

B_{p-q}

5.28. Найти температуру бесконечного кругового цилиндра $0 \le r \le r_0$, если его поверхность поддерживается при нулевой температуре, а нач. темп. равна $u|_{t=0} = u_0\left(1 - \frac{r^2}{r_0^2}\right)$.

$$\begin{cases} u|_{t=0} = u_0\left(1 - \frac{\rho^2}{\rho_0^2}\right) \\ u|_{\rho=\rho_0} = 0 \quad (г.у) \end{cases}$$

$$u_t = a^2 \Delta u$$

$u = u(\rho, t)$. Температура зависит от радиуса и времени

$$u_t = a^2 \frac{1}{\rho} \frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right), \quad u = u(\rho, t)$$

$u = R(\rho) \cdot T(t)$ – ищем решение в виде

$$R(\rho) \cdot T' = a^2 \frac{1}{\rho} \frac{\partial}{\partial \rho}(\rho R') \cdot T$$

$$R \cdot T' = \frac{a^2}{\rho} \frac{\partial}{\partial} T \cdot R''$$

$$\frac{T'}{T} = \frac{a^2}{\rho} \frac{R''}{R}$$

$$\begin{cases} R'' + \lambda R = 0 \\ \cancel{T' + \frac{\rho}{a^2} \lambda T = 0} \end{cases} \qquad \begin{cases} T' + a^2 \lambda T = 0 \\ R'' + \rho \lambda R = 0 \end{cases}$$

$$\begin{cases} T' + a^2\lambda T = 0 \\ \frac{1}{\rho}\left(R' + \rho R''\right) + \lambda R = 0 \end{cases} \qquad R(\rho_0) = 0$$

$$R(\rho_0)T(t) = 0, \quad R(0) < \infty$$

$$\begin{cases} \frac{1}{\rho}R' + R'' + \lambda R = 0 \quad | \rho^2 \\ R(\rho_0) = 0 \end{cases}$$

$$\rho^2 \frac{\partial^2 R}{\partial \rho^2} + \rho \frac{\partial R}{\partial \rho} + \lambda \rho^2 R = 0 \qquad R(0) = \infty \quad R(\rho_0) = 0$$

$$0 \le \rho \le \rho_0$$

1) $\lambda \le 0$ $\lambda > 0$

$$x^2 y''(x) + x y'(x) + (x^2 ...$$

$\nu = 0$

$$\sqrt{\lambda}\rho = x, \quad R(\rho) = y(x)$$

$$x^2 \frac{d^2}{dx^2} y(x) + x\frac{d}{dx} y(x) + x^2 y(x) = 0$$ – урав. ур. о-ва нул.

$$y(x) = AJ_0(x) + BN_0(x)$$

Из усл. огр. $|y(0)| < \infty$, $B = 0$

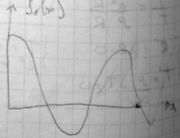
$$y(x) = AJ_0(x)$$

Бессель

$$R(\rho) = y(x) = AJ_0(x) = AJ_0(\sqrt{\lambda}\rho)$$

$$R(\rho_0) = 0$$

$$J_0(\sqrt{\lambda}\rho_0) = 0$$



$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{2}\nu - \frac{\pi}{4}\right)$$

$$J_\nu(x)=\sum_{k=0} \frac{\ldots}{k!\,\Gamma(k+\nu+1)}$$

$$J_0(x)=1-\left(\frac{x}{2}\right)^2+O(x^4)$$

$$J_0(\mu_m^{(0)})=$$

$$\mu_m^{(0)}>0,\ m=1,2,3$$

$$\sqrt{\lambda}=\frac{\mu_m^{(0)}}{\rho_0} \qquad R_m(\rho)=A\,J_0\left(\frac{\mu_m^{(0)}\rho}{\rho_0}\right)$$

$$T_m(t)=B_m e^{-\left(\frac{\mu_m^{(0)}}{\rho_0}a\right)^2 t}$$

$$u_m(\rho,t)=C_m e^{-\left(\frac{\mu_m^{(0)}}{\rho_0}a\right)^2 t} J_0\left(\frac{\mu_m^{(0)}}{\rho_0}\rho\right)$$

$$u(\rho,t)=\sum_{m=1}^{+\infty} u_m(\rho,t)$$

$$u(\rho,t)=\sum_{m=1}^{+\infty} C_m J_0\left(\frac{\mu_m^{(0)}}{\rho_0}\rho\right) e^{-\left(\frac{\mu_m^{(0)}}{\rho_0}a\right)^2 t}$$

$$u(\rho,0)=\sum_{m=1}^{+\infty} C_m J_0\left(\frac{\mu_m^{(0)}}{\rho_0}\rho\right)=u_0\left(1-\left(\frac{\rho}{\rho_0}\right)^2\right)\Bigg|\cdot\rho J_0\left(\frac{\mu_n^{(0)}}{\rho_0}\rho\right)$$

$$\left\{J_0\left(\frac{\mu_m^{(0)}}{\rho_0}\rho\right),\ m=1,2,\ldots\right\} \qquad 0\le\rho\le\rho_0$$

$$\sum_{m=1}^{+\infty} C_m \int_0^{\rho_0}\rho\, J_0\left(\frac{\mu_m^{(0)}}{\rho_0}\rho\right) J_0\left(\frac{\mu_n^{(0)}}{\rho_0}\rho\right)d\rho=$$

$$=u_0\int_0^{\rho_0}\rho\left(1-\left(\frac{\rho}{\rho_0}\right)^2\right) J_0\left(\frac{\mu_n^{(0)}}{\rho_0}\rho\right)d\rho$$

$$\rho/\rho_0=x$$

$$\rho_0^2\int_0^1 x\,J_0(\mu_m^{(0)}x)\,J_0(\mu_n^{(0)}x)\,dx =$$ [illegible]

$= \rho_0^2 \delta_{nm} \frac{1}{2} \left(J_0' (\mu_n^{(0)}) \right)^2$

$$\left| \int_0^1 dx\, x J_\nu^2 (\lambda x) = \frac{1}{2} \left\{ [J_\nu'(\lambda)]^2 + \left(1 - \frac{\nu^2}{\lambda^2}\right) J_\nu^2(\lambda) \right\} \right.$$

$$C_n = \frac{2}{\rho_0^2} \frac{u_0}{J_1^2(\mu_n^{(0)})} \underbrace{\int_0^{\rho_0} \rho \left(1 - \frac{\rho^2}{\rho_0}\right) J_0\left(\mu_n^0 \frac{\rho}{\rho_0}\right) d\rho}_{I}$$

$$\mu_n^{(0)} \frac{\rho}{\rho_0} = x$$

$$I = \int_0^{\rho_0} \rho \left(1 - \frac{\rho^2}{\rho_0}\right) J_0\left(\mu_n^{(0)} \frac{\rho}{\rho_0}\right) d\rho =$$

$$= \int_0^{\mu_n^{(0)}} x \left(1 - \left(\frac{x}{\mu_n^{(0)}}\right)^2\right) J_0(x)\, dx$$

$$x J_0(x) = \frac{d}{dx}(x J_1(x))$$

$$\int x J_0(x) dx = \left| \frac{d}{dx}[x^\nu J_\nu(x)] = x^\nu J_{\nu-1}(x) \right| = x J_1(x)$$

$$\cancel{\int x^3 J_0(x) dx}$$

$$\int x^3 J_0(x) = \int x^2 \frac{d}{dx}(x J_1(x))\, dx = x^3 J_1(x) - 2 \int x^2 J_1(x) dx =$$

$$= x^3 J_1(x) - 2x^2 J_2(x) + C_1$$

$$I = \mu_n^{(0)} J_1(\mu_n^{(0)}) - \mu_n^{(0)} J_1(\mu_n^{(0)}) + 2 J_2(\mu_n^{(0)})$$

$$C_n = \frac{2u_0}{\rho_0^2 J_1^2(\mu_n^{(0)})} \frac{\rho_0^2}{(\mu_n^{(0)})^2} \cancel{= 2 J_2(\mu_n^{(0)})} =$$

$$J_{\nu+1}(x) + J_{\nu-1}(x) = \frac{2\nu}{x} J_\nu(x)$$

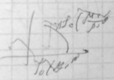
$$J_2(\mu_n^{(0)}) = \frac{2}{\mu_n^{(0)}} J_1(\mu_n^{(0)})$$

$$\frac{4u_0 J_2(\mu_n^{(0)})}{(\mu_n^{(0)})^2 J_1^2(\mu_n^{(0)})} \quad \sum_{m=1}^{\infty} C_m J_0\left(\frac{\mu_m^{(0)}}{\rho_0}\rho\right) e^{-\left(\frac{\mu_m^{(0)}}{\rho_0}a\right)^2 t} = u(\rho,t)$$

$$C_m = \frac{8u_0}{(\mu_n^{(0)})^3 J_1(\mu_n^{(0)})}$$

$$u(\rho,t) = 8u_0 \sum_{n=1}^{\infty} \frac{1}{(\mu_n^{(0)})^3 J_1(\mu_n^{(0)})} \exp\left(-\left(\frac{\mu_n^{(0)} a}{\rho_0}\right)^2 t\right) J_0\left(\mu_n^{(0)}\frac{\rho}{\rho_0}\right)$$

$$\tau = \frac{\rho_0^2}{a^2[\mu_1^{(0)}]^2}, \quad \mu_1^{(0)} \approx 2{,}4$$



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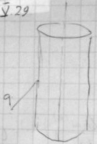

$u\big|_{t=0} = u_0$

$u - ?$

$u_t = a^2 \Delta u$

$\vec{Q} = -\varkappa \nabla u$. – поток тепла

$-Q \vec{e_\rho} = q$

$\vec{e_\rho} = \vec{\rho}/\rho$

$\nabla u = \left(\frac{\partial u}{\partial \rho}\right)\vec{e_\rho} + \left(\frac{\partial u}{\rho \partial \varphi}\right)\vec{e_\varphi} + \left(\frac{\partial u}{\partial z}\right)\vec{e_z}$

$-\varkappa \nabla u \frac{\vec{\rho}}{\rho} = q = const$

$\left(\frac{\partial u}{\partial \rho}\right)\Big|_{\rho=\rho_0} = \frac{q}{\varkappa}$

$\Delta u = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2}$

$$\begin{cases} \frac{\partial u_1}{\partial \rho} = \frac{q}{\varkappa} = \\ u(\rho, 0) \end{cases}$$

$u_2(\rho, t) = f(\rho)t + g(\rho)$

получим д/р $u_{2t} = f(\rho)$.

$\Delta u_2 = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} = f''(\rho)t + g''(\rho) + \frac{f'(\rho)}{\rho}t + \frac{g'(\rho)}{\rho}$

$f(\rho) = a^2 f''(\rho)t + a^2 g(\rho) + a^2 \frac{f'(\rho)}{\rho}t + a^2\frac{g'(\rho)}{\rho}$

$$\begin{cases} f u^2\left(f''(\rho) + \dfrac{f'(\rho)}{\rho}\right) = 0 \rightarrow \\ u^2 g''(\rho) + u^2 \dfrac{g'(\rho)}{\rho} - f(\rho) = 0 \end{cases}$$

$$a^2 \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dg}{d\rho}\right) = B$$

$$\rho \frac{dg}{d\rho} = \frac{B}{2a^2}\rho^2 + D$$

$$g = \frac{B\rho^2}{4a^2} + \cancel{D}\ln\rho + F$$

$$|g(0)| < \infty \rightarrow D = 0$$

$$g'(\rho_0) = \frac{q}{\varkappa}$$

$$\frac{q}{\varkappa} = \frac{B\rho_0}{2a^2}, \quad B = \frac{q\, 2a^2}{\varkappa \rho_0}$$

$$u_2(\rho, t) = \frac{2qa^2}{\varkappa\rho_0}t + \frac{q}{2\varkappa\rho_0}\rho^2 + F;$$

$$u(\rho, t) = u_{\text{част}}(\rho, t) + v(\rho, t)$$

$$v(\rho, t) = a^2 \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d\varphi}{d\rho}\right);$$

$$\left.\frac{dv}{d\rho}\right|_{\rho=\rho_0} = 0$$

$$v = R(\rho)T(t)$$

$$R(\rho)\,T'(t) = a^2 \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho \frac{\partial R(\rho)}{\partial\rho}\right)T(t)$$

$$\frac{\ldots}{R(\rho)T(t)a^2} = \frac{\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial R}{\partial \rho}\right)(a}{R(\rho)T(t)} = -\lambda$$

$$\begin{cases} T'(t) + \lambda a^2 T(t) = 0 \\ \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial R(\rho)}{\partial \rho}\right) + \lambda R(\rho) = 0 \end{cases}$$

$T(t) = C\exp(-\lambda a^2 t)$

1) $\lambda = 0$ (2) $\lambda > 0$

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR(\rho)}{d\rho}\right) = 0$$

$R = const$

$V = V_0,\ \lambda = 0$

$\rho^2 R'' + \rho R' + (\sqrt{\lambda}\rho)^2 R = 0$

Заміна $\sqrt{\lambda}\rho = x;\ R(\rho) = y(x)$

$x^2 y'' + xy' + x^2 y = 0$

$y(x) = AJ_0(x) + BN_0(x);$

$R(\rho) = AJ_0(\sqrt{\lambda}\rho) + BN_0(\sqrt{\lambda}\rho);$

$0 \le \rho \le \rho_0$

$R'(\rho_0) = 0\ \sqrt{\lambda}AJ_0'(\sqrt{\lambda}\rho) = 0 \Rightarrow$

$\{J_0'(x) = -J_1(x)\} \Rightarrow J_1(\sqrt{\lambda}\rho) = 0 \Rightarrow$

$\Rightarrow$

$\Rightarrow \sqrt{\lambda}\rho_0 = \mu_m^{(1)}, \; m=1,2 \qquad \sqrt{\lambda} = \frac{\mu_m^{(1)}}{\rho_0} \Rightarrow$

$\Rightarrow R_m(\rho) = A_m J_0\left(\frac{\mu_m^{(1)}}{\rho_0}\rho\right), \; T_m = B_m e^{-\left(\frac{\mu_m^{(1)}}{\rho_0}a\right)^2 t}$

$$v(\rho,t) = \sum_{m=1}^{+\infty} C_m J_0\left(\frac{\mu_m^{(1)}}{\rho_m}\rho\right)\exp\left\{-\left(\frac{\mu_m^{(1)}}{\rho_0}a\right)^2 t\right\} + C_0$$

$$v(\rho,0) = \sum_{m=1}^{+\infty} C_m J_0\left(\frac{\mu_m^{(1)}}{\rho_0}\rho\right) + C_0 = u_0 - \frac{t}{2\varkappa\rho_0}\rho^2\Big|_{\rho}.$$

Такие набор

$$\left\{1, J_0\left(\frac{\mu_m^{(1)}}{\rho_0}\rho\right), m=1\right)$$

$$\int_0^{\rho_0} d\rho\, \rho J_0\left(\frac{\mu_m^{(1)}}{\rho_0}\rho\right)\cdot 1 = \left|\frac{\mu_m^{(1)}}{\rho_0}\rho = x\right| = \left(\frac{\rho_0}{\mu_m^{(1)}}\right)^2 \int_0^{\mu_m^{(1)}} dx\, x J_0(x) =$$

$$= \left(\frac{\rho_0}{\mu_m^{(1)}}\right)^2 x J_1(x)\Big|_0^{\mu_m^{(1)}} = 0$$

$$\int_0^1 dx\, x\, J_1/dx$$

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$u_t = a^2 \Delta u.$

гр. условие

$u\big|_{r=r_0} = 0 \qquad u\big|_{t=0} = f(r,\theta,\varphi) = u_0 \frac{r}{r_0} \sin\theta\cos\varphi$

Методом разделения переменных

Ищем решение в виде $u = T(t)\,R(r)\,Y(\theta,\varphi)$

$$T'(t)R(r)Y(\theta,\varphi) = a^2\left(T\left(\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)Y + \Delta_{\theta,\varphi}Y(\theta,\varphi)\cdot R\right)\right)$$

$$\frac{T'(t)}{a^2T} = \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)Y + \Delta_{\theta,\varphi}Y(\theta,\varphi)R\right]\frac{1}{RY}$$

$= -\mu.$

$$-\lambda = \frac{R\,\Delta_{\theta,\varphi}Y(\theta,\varphi)}{RY.}$$

$$-\mu = \frac{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)Y}{YR} + \frac{1}{r^2}(-\lambda)$$

$$-\mu = \frac{1}{r^2}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\cdot\frac{1}{R} - \frac{1}{r^2}\lambda \quad \Big| R.$$

$$\begin{cases} \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + R\left(\mu - \frac{1}{r^2}\lambda\right) = 0, & 0\le r\le r_0 \\ T'(t) + \mu a^2 T = 0, & t>0 \\ \Delta_{\theta,\varphi}Y(\theta,\varphi) + \lambda Y = 0 & \varphi\in[0,2\pi] \end{cases}$$

$Y_{mn}(\theta,\varphi) = Y$

Ищем особенности не ни с реше

$\lambda = n(n+1)$

Для сферических функций

Сфер/ич. остается до нужно [illegible]

$$Y_{nm}(\theta, \varphi) = \begin{cases} P_{n|m|}(\cos\theta)\cos m\varphi, & -n \le m < 0 \\ P_{nm}(\cos\theta)\sin m\varphi & 0 \le m \le n \end{cases}$$

$$\boxed{\begin{array}{l} \mu > 0 \\ \lambda = n(n+1) \end{array}}$$

$$z^2 \frac{d^2 R}{dz^2} + 2z\frac{dR}{dz} + (\mu z^2 - n(n-1))R = 0$$

$$x = \sqrt{\mu}\, z$$

$$R(z) = y(x)$$

$$x^2 y'' + 2xy' + [x^2 - n(n-1)]y = 0$$ — Лапласа ур.

1) $y(x) = \dfrac{z(x)}{\sqrt{x}}$ $\qquad$ $z(x) = x^{\lambda} y(x)$; $x = t^{p}$

$$y'' = \left(\frac{z'\sqrt{x} - \frac{1}{2\sqrt{x}} z}{x} \right)' = z'' x^{-1/2} - \frac{1}{2} z' x^{-3/2} - z' \frac{1}{2} x^{-3/2} + \frac{3}{4} z x^{-5/2} =$$

$$= z'' x^{-1/2} - z' x^{-3/2} + \frac{3}{4} z x^{-5/2}.$$

$$y' = \frac{z'}{\sqrt{x}} - \frac{1}{2}\frac{z}{x^{3/2}}$$

Подставим в у/ние Лапласа

$$x^2 \left[z'' x^{-1/2} - z' x^{-3/2} + \frac{3}{4} x^{-5/2} z \right] + 2x\left(z' x^{-1/2} - \frac{1}{2} z x^{-3/2}\right) +$$

$$+ [x^2 - n(n+1)] z x^{-1/2} = 0$$

$$x^{1/2}\cdot\Big| z''x^{3/2}+z'\left[-x^{1/2}+2x^{1/2}\right]+z\left[\frac{3}{4}x^{-1/2}-x^{-1/2}+\right.$$

$$\left.+x^{3/2}-n(n+1)^{-1/2}\right]=0$$

$$x^2z''+xz'+z\Big[\underbrace{-\frac{1}{4}-n(n+1)}_{=-(n+\frac{1}{2})^2}+x^2\Big]=0$$

$$x^2z''+xz'+z\left[x^2-\nu^2\right]=0 \quad \text{— Бес. ур-е}$$

$$\nu=n+\frac{1}{2}$$

Ищем решение в виде

$$z(x)=A\,J_{n+\frac{1}{2}}(x)+B\,J_{-n-\frac{1}{2}}(x)$$

Тогда

$$R(r)=\frac{1}{\sqrt{r}}\left[A\,J_{n+\frac{1}{2}}(\sqrt{\mu}\,r)+B\,J_{-n-\frac{1}{2}}(\sqrt{\mu}\,r)\right)$$

1) $|R(0)|=\infty$, $R(r_0)=0$, $r\in[0,r_0]$

$B=0$ $\quad (J_{-n}(x)=(-1)^n J_n(x))$

2) $R(r_0)=0$

$$\frac{A}{\sqrt{r_0}}J_{n+\frac{1}{2}}(\sqrt{\mu}\,r_0)=0$$

$$\sqrt{\mu}\,r_0=\mu_m^{(n+\frac{1}{2})}$$

$$R_{n,p}(r)=\frac{A}{\sqrt{r}}J_{n+\frac{1}{2}}\left(\frac{\mu_p^{(n+\frac{1}{2})}}{r_0}r\right)$$

$$u(r,\theta,\varphi,t)=\sum_{n=0}^{+\infty}\sum_{m=-n}^{n}\sum_{p=1}^{\infty}A_{nmp}\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}\left(\frac{\mu^{n+\frac{1}{2}}}{r_0}r\right)Y_n^m(\theta\ldots$$

$$e^{-\left(a \frac{\mu_p^{n+\frac12}}{r_0}\right)^2 t}$$

$$u\Big|_{t=0} = f(r,\theta,\varphi)$$

Для определения постоянных коэффициентов.

$$f(r,\theta,\varphi) = \sum_{nmp} A_{nmp} \frac{1}{\sqrt{r}} J_{n+1/2}\left(\frac{\mu_p^{n+\frac12}}{r_0} r\right) Y_{nm}(\theta,\varphi) \Big|\, Y_{kl}(\theta,\varphi)\ \sin\theta\, d\theta$$

$$\left(\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi\, f(r,\theta,\varphi) Y_{kl}\right) = \sum_{p=1}^{+\infty} A_{klp} \frac{1}{\sqrt{r}} J_{k+\frac12}\left(\frac{\mu_p^{k+\frac12}}{r_0} r\right)$$

$$\|Y_{kl}\|^2 \Big|\, r^{3/2} J_{k+\frac12}\left(\frac{\mu_s^{(k-\frac12)}}{r_0} r\right) = e^{-\left(a\mu_p^{(n-\frac12)}/r_0\right)^2 t}$$

$$\int_0^{r_0} dr\, r^{3/2} J_{k+\frac12}\left(\frac{\mu_s^{k+\frac12}}{r_0} r\right)\left(\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\theta\, \sin\theta \int d\varphi\, f(r,\theta,\varphi) Y_{kl}\right) =$$

$$= A_{kls} \|Y_{kl}\|^2 \left\| J_{k+\frac12}\left(\frac{\mu_s^{k+\frac12} r}{r_0}\right)\right\|^2$$

Посчитаем интеграл.

$$\int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\varphi\, Y_{kl}(\theta,\varphi) \cdot \sin\theta\cos\varphi\, P_1^1(\cos\theta) \cdot \cos\varphi$$

$$(Y_{kl}, Y_{1-1}) = \frac{(1+1)!}{0!} \frac{2\pi(1)}{2+1} \delta_{k1} \delta_{l,-1} = \frac{4\pi}{3} \delta_{k1} \delta_{l,-1}$$

$$\int_0^{r_0} dr\, r^{3/2} J_{3/2}\left(\frac{\mu_1}{r_0} r\right) = \left(\frac{r_0}{\mu_s^{3/2}}\right)^{7/2} \int_0^{\mu} dx\, x^{3/2} J_{3/2}(x) =$$

$$= \left(\frac{r_0}{\mu_s}\right)^{3/2} \int_0^{\mu} dx\, x^{3/2} J_{3/2}(x) = \left(\frac{r_0}{\mu_s}\right)^{1/2} \mu^{3/2} J_{3/2}(\mu)$$

$$= \,!$$

Д/з №1:

I.2. $u_{xx} + x u_{yy} = 0$

$a = 1,\ c = x,\ b = 0$

$\Delta = b^2 - ac = 0 - x = -x < 0$

$\Delta > 0$ – Гиперболич.
$\Delta < 0$ – эллиптическ.
$\Delta = 0$ – параболическ.

в области $x < 0$ – гиперболично
в области $x > 0$ – эллиптично

а) Область гиперболичности

$$\frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a}, \quad \frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a}$$

$$\frac{dy}{dx} = \frac{0 + \sqrt{0 - 1 \cdot x}}{1} = \sqrt{-x}, \quad \frac{dy}{dx} = -\sqrt{-x}$$

$$dy = \sqrt{-x}\, dx$$

$$y = \int \sqrt{-x}\, dx = -\int \sqrt{-x}\, d(-x) = -\frac{2}{3}(-x)^{3/2} + C$$

1) $C = y + \frac{2}{3}(-x)^{3/2}$ — общие интегралы

2) $y = -\int \sqrt{-x}\, dx = \frac{2}{3}(-x)^{3/2} + C,\ C = y - \frac{2}{3}(-x)^{3/2}$

Произведем замену

$$\xi = y + \frac{2}{3}(-x)^{3/2}$$
$$\eta = y - \frac{2}{3}(-x)^{3/2}$$

$$\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial y}, \frac{\partial \eta}{\partial x} - ?$$

1. $\frac{\partial \xi}{\partial x} = \frac{2}{3} \cdot \frac{3}{2}\sqrt{-x} = \sqrt{-x}$ | $\frac{\partial \eta}{\partial x} = -\sqrt{-x}$

$\frac{\partial \xi}{\partial y} = 1$ | $\frac{\partial \eta}{\partial y} = 1$

2. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2} - ?$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi}\sqrt{-x} - \frac{\partial u}{\partial \eta}\sqrt{-x} =$$

$$= \sqrt{-x}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right) = \sqrt{-x}\,(u_\xi - u_\eta)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} = u_\xi + u_\eta$$

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\right)\cdot\sqrt{-x} + \frac{\partial u}{\partial \xi}\cdot\frac{1}{2\sqrt{-x}} - \sqrt{-x}\left(\frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\right.$$

$$\left. + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial x}\right) - \frac{1}{2\sqrt{-x}}\frac{\partial u}{\partial \eta} = \sqrt{-x}\,(u_{\xi\xi}\sqrt{-x} + u_{\xi\eta}(-\sqrt{-x}) +$$

$$+ u_{\eta\eta}\sqrt{-x} - u_{\eta\xi}\sqrt{-x}) + \frac{1}{2\sqrt{-x}}(u_\xi - u_\eta) =$$

$$= (-x u_{\xi\xi} + x u_{\xi\eta} - u_{\eta\eta}x + x u_{\eta\xi}) + \frac{1}{2\sqrt{-x}}(u_\xi - u_\eta)$$

$$= x(u_{\xi\eta} + u_{\eta\xi} - u_{\xi\xi} - u_{\eta\eta}) + \frac{1}{2\sqrt{-x}}(u_\xi - u_\eta)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(u_\xi + u_\eta) = \frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial y}$$

$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

Подставим в исх. уравнение $u_{xx} + x u_{yy} = 0$

$$x(2u_{\xi\eta} - u_{\xi\xi} - u_{\eta\eta}) + \frac{1}{2\sqrt{-x}}(u_\xi - u_\eta) + x(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) = 0$$

$$\boxed{4x u_{\xi\eta} + \frac{1}{2\sqrt{-x}}(u_\xi - u_\eta) = 0}$$

б) В области эллиптичности $\Delta < 0$

Замена $x' = y$

$$y' = \frac{2}{3}(x)^{3/2}.$$

I-3. $u_{xx} + y u_{yy} = 0$

$a = 1,\ c = y,\ b = 0$

$\Delta = b^2 - ac = 0 - y = -y$

в обл. гиперболичности $y < 0 \Rightarrow \Delta > 0$

в обл. эллиптичности $y > 0 \Rightarrow \Delta < 0$

а) Гипербол. обл.

$$\frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a} \qquad \frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a}$$

$$\frac{dy}{dx} = \frac{0 + \sqrt{0 - 1 \cdot y}}{1} = \sqrt{-y} \qquad \frac{dy}{dx} = -\sqrt{-y}$$

$$dy = \sqrt{-y}\, dx$$

$$x = \int \frac{1}{\sqrt{-y}} dy = -\int \frac{1}{\sqrt{-y}} d(-y) = -2\sqrt{-y} + C$$

$C = x + 2\sqrt{-y}$ — ξ — Замены переменных

$C = x - 2\sqrt{-y}$ — η

1. $\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$ — ?

$$\frac{\partial \xi}{\partial x} = 1 \qquad \frac{\partial \eta}{\partial x} = 1$$

$$\frac{\partial \xi}{\partial y} = \frac{1}{\sqrt{-y}} \qquad \frac{\partial \eta}{\partial y} = -\frac{1}{\sqrt{-y}}$$

2. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}$ — ?

1) $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} = u_\xi + u_\eta$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{\sqrt{-y}} u_\xi - \frac{1}{\sqrt{-y}} u_\eta$$

$$2)\ \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\partial u_\xi + u_\eta\right) = u_{\xi\xi}\frac{\partial \xi}{\partial x} + u_{\xi\eta}\frac{\partial \eta}{\partial x} + u_{\eta\eta}\frac{\partial \eta}{\partial x} +$$

$$+ u_{\eta\xi}\frac{\partial \xi}{\partial x} = u_{\xi\xi} + u_{\eta\eta} + \frac{1}{\sqrt{-y}}u_{\eta\xi} - \frac{1}{\sqrt{-y}}u_{\eta\xi} =$$

$$= u_{\xi\xi} + u_{\eta\eta} + u_{\eta\xi}\left(\frac{1}{\sqrt{-y}} - \frac{1}{\sqrt{-y}}\right) = u_{\xi\xi} + u_{\eta\eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{-y}}u_\xi - \frac{1}{\sqrt{-y}}u_\eta\right) = -\frac{1}{2(-y)^{3/2}}\left(u_\xi + \frac{1}{\sqrt{-y}}\left(u_{\xi\xi}\frac{\partial \xi}{\partial y}\right.\right.$$

$$\left.+ u_{\xi\eta}\frac{\partial \eta}{\partial y}\right) + \frac{1}{2(-y)^{3/2}}u_\eta - \frac{1}{\sqrt{-y}}\left(u_{\eta\eta}\frac{\partial \eta}{\partial y} + u_{\eta\xi}\frac{\partial \xi}{\partial y}\right) =$$

$$= -\frac{1}{2(-y)^{3/2}}u_\xi + \frac{1}{\sqrt{-y}}\cdot\frac{1}{\sqrt{-y}}u_{\xi\xi} + \frac{1}{\sqrt{-y}}u_{\xi\eta}\cdot\frac{1}{\sqrt{-y}} + \frac{1}{(2y)^{3/2}}u_\eta + \frac{1}{\sqrt{-y}}u_{\eta\eta}\frac{1}{\sqrt{-y}}$$

$$-\frac{1}{\sqrt{-y}}\cdot\frac{1}{\sqrt{-y}}u_{\eta\xi} = -\frac{1}{2(-y)^{3/2}}u_\xi - \frac{1}{y}u_{\xi\xi} + u_{\xi\eta}\frac{1}{y} + \frac{1}{2y^{3/2}}u_\eta - \frac{1}{y}u_{\eta\eta}$$

$$+ \frac{1}{y}u_{\eta\xi}$$

Подставим в исх. ур.

$$u_{\xi\xi} + u_{\eta\eta} - \frac{y}{2y^{3/2}}u_\xi - \frac{y}{y}u_{\xi\xi} + \frac{y}{y}u_{\xi\eta} + \frac{y}{2y^{3/2}}u_\eta - \frac{y}{y}u_{\eta\eta} +$$

$$+\frac{y}{y}u_{\eta\xi} = 0$$

$$+\frac{1}{2\sqrt{-y}}u_\xi + u_{\xi\eta} + \frac{1}{2\sqrt{-y}}u_\eta + u_{\xi\eta} = 0$$

$$2u_{\xi\eta} + \frac{1}{2\sqrt{-y}}(u_\eta + u_\xi) = 0$$

$$\boxed{u_{\xi\eta} + \frac{1}{2(\xi-\eta)}(u_\xi + u_\eta) = 0}$$

б) Область эллиптичности

Замена $\xi' = x$, $\eta' = 2\sqrt{y}$

1) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi'}\frac{\partial \xi'}{\partial x} + \frac{\partial u}{\partial \eta'}\frac{\partial \eta'}{\partial x} = u_{\xi'}$

$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi'}\frac{\partial \xi'}{\partial y} + \frac{\partial u}{\partial \eta'}\frac{\partial \eta'}{\partial y} = \frac{2\cdot u_{\eta'}}{2\sqrt{y}} = \frac{1}{\sqrt{y}} u_{\eta'}$

2) $\frac{\partial^2 u}{\partial x^2} = u_{\xi'\xi'} = u_{xx}$

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{y}}\cdot u_{\eta'}\right) = -\frac{1}{2y^{3/2}} u_{\eta'} + \frac{1}{\sqrt{y}} u_{\eta'\eta'}\cdot\frac{1}{\sqrt{y}} =$

$= -\frac{1}{2y^{3/2}} u_{\eta'} + \frac{1}{y} u_{\eta'\eta'} = u_{yy}.$

Подставим в исх. ур. $u_{xx} + y u_{yy} = 0$

$u_{\xi'\xi'} - \frac{y}{2y^{3/2}} u_{\eta'} + \frac{y}{y} u_{\eta'\eta'} = 0$

$u_{\xi'\xi'} - \cancel{y}\frac{u_{\eta'}}{2\sqrt{y}} + u_{\eta'\eta'} = 0$

$$\boxed{u_{\xi'\xi'} - \frac{u_{\eta'}}{\eta'} + u_{\eta'\eta'} = 0}$$

I.S. $y u_{xx} + x u_{yy} = 0$

$a = y,\ b = 0,\ c = x$

$\Delta = 0 - yx = -yx$

а) Рассмотрим обл. гиперболичности $\Delta > 0$, ~~x<0 & y<0~~

$\frac{dy}{dx} = \frac{b + \sqrt{b^2 - ac}}{a}$ $\quad \frac{dy}{dx} = \frac{b - \sqrt{b^2 - ac}}{a}$ $\quad \sqrt{y}$

$\frac{dy}{dx} = \frac{\sqrt{xy}}{y} = \frac{\sqrt{x}}{\sqrt{y}}$ $\quad \frac{dy}{dx} = -\frac{\sqrt{x}}{\sqrt{y}}$, $\quad y(dy)^2 = -x\,dx^2$

I 15.

$$y^2 u_{xx} + 2xy\, u_{xy} + x^2 u_{yy} = 0$$

$$a_{11} = y^2,\ a_{12} = xy,\ a_{22} = x^2$$

$\Delta = a_{12}^2 - a_{11}a_{22} = x^2y^2 - x^2y^2 = 0$ Уравнение повыду параболического типа

$$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} = \frac{xy \pm \sqrt{x^2y^2 - x^2y^2}}{y^2} =$$

$= \frac{x}{y}$ - одно семейство хар-тик

$$dy \cdot y = dx \cdot x$$

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C\ ;\ C = \frac{y^2}{2} - \frac{x^2}{2} \text{ or } C = \frac{x^2 - y^2}{2}.$$

$$\xi = \frac{x^2 + y^2}{2}\ ;\ \eta = \frac{x^2 - y^2}{2}.$$

1) $\frac{\partial \xi}{\partial x}, \frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial x}, \frac{\partial \eta}{\partial y}$ - ?

$$\frac{\partial \xi}{\partial x} = x \qquad \frac{\partial \eta}{\partial x} = x$$

$$\frac{\partial \xi}{\partial y} = y \qquad \frac{\partial \eta}{\partial y} = -y$$

2) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x \partial y}$ - ?

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = (u_\xi + u_\eta)x$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} = (u_\xi - u_\eta)y$$

$$+ x(u_{\eta\eta} \cdot x + u_{\eta\xi} \cdot x)$$

$$\frac{\partial^2 u}{\partial x^2} = (u_\xi + u_\eta) + x(u_{\xi\xi} \cdot x + u_{\xi\eta} \cdot x) = (u_\xi + u_\eta) + x^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$\frac{\partial^2 u}{\partial y^2} = (u_\xi - u_\eta) + y\,(u_{\xi\xi}\cdot y + u_{\xi\eta}\cdot y - u_{\eta\eta}\cdot y - u_{\eta\xi}\cdot y) =$$

$$= (u_\xi - u_\eta) + y^2 (u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta})$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left((u_\xi - u_\eta)\,y\right) = y\,(u_{\xi\xi}\cdot x + u_{\xi\eta}\cdot x - u_{\eta\eta}\cdot x - \ldots$$

$$= yx\,(u_{\xi\xi} - u_{\eta\eta})$$

Подставим в исх. уравнение

$$y^2\left((u_\xi + u_\eta) + x^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})\right) + 2xy\cdot yx\,(u_{\xi\xi} - \ldots$$

$$+ x^2\left((u_\xi - u_\eta) + y^2(u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta})\right) = 0$$

$$y^2(u_\xi + u_\eta) + x^2(u_\xi - u_\eta) + 2x^2y^2(u_{\xi\xi} - u_{\eta\eta})$$

$$\boxed{\frac{\partial^2 u}{\partial \xi^2} + \frac{\xi}{2(\xi^2 - \eta^2)}\frac{\partial u}{\partial \xi} - \frac{\eta}{2(\xi^2 - \eta^2)}\frac{\partial u}{\partial \eta} = 0}$$

I 16. $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy} = 0$

$\Delta = x^2y^2 - x^2y^2 = 0.$

$$\frac{dy}{dx} = \frac{a_{12} \pm \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} = \frac{xy}{x^2} = \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x}$$

$$\frac{dy}{y} = \frac{dx}{x}$$

$$\ln y = \ln x + C$$

$$C_1 = \ln\frac{y}{x}$$

$C_1 = \frac{y}{x} = \xi.$ $\eta = y$ - η можно выбрать любой

1) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\partial y}$ - ?

$$\frac{\partial u}{\partial x} = u_\xi\left(-\frac{y}{x^2}\right)$$

$$\frac{\partial^2 u}{\partial x^2} = -\left(-\frac{2y}{x^3}u_\xi + \frac{y}{x^2}u_{\xi\xi}\cdot\frac{y}{x^2}\right) = \frac{2y}{x^3}u_\xi + \frac{y^2}{x^4}u_{\xi\xi}.$$

$$\frac{\partial u}{\partial y} = u_\xi\cdot\frac{1}{x} + u_\eta$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}\left(u_{\xi\xi}\frac{1}{x} + u_{\xi\eta}\right) + u_{\eta\eta} + u_{\eta\xi}\cdot\frac{1}{x}$$

$$\frac{\partial^2 u}{\partial x\partial y} = -\left(\left(u_{\xi\xi}\cdot\frac{1}{x} + u_{\xi\eta}\right)\frac{y}{x^2} + \frac{u_\xi}{x^2}\right)$$

$$x^2\left(\frac{2y}{x^3}u_\xi - \frac{y^2}{x^4}u_{\xi\xi}\right) + 2xy\left(-u_{\xi\xi}\frac{y}{x^3} - u_{\xi\eta}\cdot\frac{y}{x^2} - \frac{u_\xi}{x^2}\right) + y^2\cdot\frac{1}{x}$$

$$\left(u_{\xi\xi}\cdot\frac{1}{x} + u_{\xi\eta} + u_{\eta\eta} + u_{\eta\xi}\cdot\frac{1}{x}\right) = 0$$

$$\cancel{\frac{2y}{x}u_\xi} - \frac{y^2}{x^2}u_{\xi\xi} - 2u_{\xi\xi}\cdot\frac{y^2}{x^2} - 2u_{\xi\eta}\frac{y^2}{x} - \cancel{\frac{2u_\xi\cdot y}{x}} + \frac{y^2}{x^2}u_{\xi\xi}$$

$$\frac{y^2}{x}u_{\xi\eta}+\frac{y^2}{x}u_{\eta\eta}+\frac{y^2}{x^2}u_{\eta\xi}=0$$

$$\frac{y^2}{x}u_{\eta\eta}=0$$

$$u_{\eta\eta}=0$$

T. 17. $4y^2u_{xx}-e^{2x}u_{yy}-4y^2u_x=0$

$$\Delta=b^2-ac=0-4y^2e^{2x}=-4y^2e^{2x}.$$

$\Delta<0$ Гиперболический тип

$$\frac{dy}{dx}=\frac{b+\sqrt{b^2-ac}}{a}=\frac{\sqrt{-4y^2e^{2x}}}{4y^2}$$

$$\frac{dy}{dx}=\frac{-\sqrt{-4y^2e^{2x}}}{4y^2}=-\frac{\sqrt{-e^{2x}y^2}}{2y^2}=-\frac{\sqrt{-e^{2x}}}{2y}$$

$$dy\cdot 2y=dx\cdot\sqrt{-e^{2x}}$$

$$y^2=-\frac{e^{2x}}{2}+C$$

$$\xi=e^x+y^2$$

$$\eta=-e^x+y^2$$

$$\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x}=u_\xi\cdot e^x-u_\eta e^x=e^x(u_\xi-u_\eta)$$

$$\frac{\partial u}{\partial y}=u_\xi\cdot 2y+u_\eta\cdot 2y=2y(u_\xi+u_\eta)$$

$$\frac{\partial^2 u}{\partial x^2} = e^x(u_\xi - u_\eta) + e^x(u_{\xi\xi} \cdot e^x - u_{\xi\eta} \cdot e^x + u_{\eta\eta} \cdot e^x - e^x u_{\xi\eta}) =$$

$$= e^x(u_\xi - u_\eta) + (u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}) e^{2x}$$

$$\frac{\partial^2 u}{\partial y^2} = 2(u_\xi + u_\eta) + 2y(u_{\xi\xi} 2y + u_{\xi\eta} \cdot 2y + u_{\eta\eta} 2y + u_{\xi\eta} 2y) =$$

$$= 2(u_\xi + u_\eta) + 4y^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta})$$

$$4y^2(e^x(u_\xi - u_\eta) + e^{2x}(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta})) - e^{2x}(2(u_\xi + u_\eta) +$$

$$+ 4y^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - 4y^2 \cdot e^x(u_\xi - u_\eta)$$

$$-4y^2 e^{2x} \cdot 2u_{\xi\eta} - 2e^{2x} u_\xi - 2e^{2x} u_\eta - 4y^2 e^{2x} \cdot 2u_{\xi\eta}$$

$$\cancel{e^{2x}} \quad -4y^2 u_{\xi\eta} - u_\xi - u_\eta - 4y^2 u_{\xi\eta} = 0$$

$$8y^2 u_{\xi\eta} = (u_\xi + u_\eta)$$

$$u_{\xi\eta} = \frac{1}{8y^2}(u_\xi + u_\eta)$$

$$\boxed{u_{\xi\eta} = \frac{1}{(\xi + \eta)^2 \cdot 8}(u_\xi + u_\eta)}$$

I-21.

$$a u_{xx} + 4a u_{xy} + a u_{yy} + b u_x + c u_y + u = 0$$

$$a\,dy^2 - 4a\,dx\,dy + a\,dx^2 = 0$$

$$(dy - 2dx)^2 - 3dx^2 = 0$$

$$(dy - 2dx - \sqrt{3}dx)(dy - 2dx + \sqrt{3}dx) = 0$$

$$dy = (2 \pm \sqrt{3})dx$$

$$y = (2 \pm \sqrt{3})x + C$$

Метод разделения переменных

П. 83

$x=0,\ x=l.$

$0 \qquad\qquad\qquad l$

н.у. $\begin{cases} u(x,0)=A\sin\frac{\pi x}{l} \\ u_t(x,0)=0 \end{cases}$ г.у. $\begin{cases} u(0,t)=0 \\ u(l,t)=0 \end{cases}$

Ищем решение в виде

$u(x,t)=X(x)T(t)$

$$u_{tt}=a^2u_{xx}$$

$$\frac{T''}{a^2T}=\frac{X''}{X}=-\lambda=const$$

$\begin{cases} X''+a^2X=0 \\ T''-\lambda a^2T=0 \end{cases}$

г.у. $\begin{matrix} u(0,t)=X(0)T(t)=0 \\ u(l,t)=X(l)T(t)=0 \end{matrix} \Rightarrow \begin{cases} X(0)=0 \\ X(l)=0 \end{cases}$

Определим знак λ.

1) $\lambda<0$.

$X(x)=C_1e^{\sqrt{|\lambda|}x}+C_2e^{-\sqrt{|\lambda|}x}$

$\begin{matrix} X(0)=C_1+C_2=0 \\ X(l)=C_1e^{\sqrt{\lambda}l}+C_2e^{-\sqrt{\lambda}l}=0 \end{matrix} \Rightarrow \begin{vmatrix} 1 & 1 \\ e^{\sqrt{\lambda}l} & e^{-\sqrt{\lambda}l} \end{vmatrix}=e^{\sqrt{\lambda}l}-e^{-\sqrt{\lambda}l}\neq 0$

$C_1=0$

$C_2=0$ Не подходит

2) $\lambda=0$

$\begin{cases} C_2=0 \\ C_1l+C_2=0 \end{cases} \Rightarrow C_1, C_2=0$ Не подходит

3) $\lambda > 0$

$X(x) = C_1 \sin(\sqrt{\lambda} x) + C_2 \cos(\sqrt{\lambda} x)$

$\begin{cases} C_2 = 0 \\ C_1 \sin(\sqrt{\lambda} l) = 0 \end{cases}$ $\underline{X_k(x) - ?}$

$C_1 \sin(\sqrt{\lambda} l) = 0$

$C_1 \neq 0$

$\sin \sqrt{\lambda} l = 0$

$\sqrt{\lambda} l = \pi k$

$\sqrt{\lambda} = \frac{\pi k}{l}$

$\lambda = \left(\frac{\pi k}{l}\right)^2$

$X_k(x) = C_1 \sin\left(\frac{\pi k}{l} x\right)$

Вычислим C_1. Для этого приведём к нормировке

$\|X\|^2 = \int_0^l C_1^2 \sin^2 \frac{\pi k}{l} x \, dx = 1.$

$\frac{C_1^2}{2} \int_0^l \left(1 - \cos 2\frac{\pi k}{l} x\right) dx = \frac{C_1^2 l}{2} - \frac{B_1^2}{2} \cdot \frac{l}{2\pi k} \sin \frac{2\pi k}{l} =$

$= \frac{C_1^2 l}{2} - \frac{C_1^2}{4\pi k} \cdot l \sin \pi k \cdot 2 = \frac{B_1^2 l}{2}$

$\frac{C_1^2 l}{2} = 1.$

$C_1^2 = \frac{2}{l}$

$C_1 = \sqrt{\frac{2}{l}}$, тогда

$X_k(x) = \sqrt{\frac{2}{l}} \sin \frac{\pi k}{l} x.$

II. $T'' + \lambda a^2 T = 0$

$$T_k'' - \left(\frac{\pi k}{l} a\right)^2 T_k = 0$$

$$T_k(t) = D_1 \sin\left(\frac{\pi k a}{l}\right)t + D_2 \cos\left(\frac{\pi k a}{l}\right)t$$

$$u(x,t) = \sum_{k=1}^{\infty} X_k(t)\, T_k(t) = \sum_{k=1}^{\infty} \sqrt{\frac{2}{l}} \sin\frac{\pi k}{l}x \cdot \left[D_1 \sin\left(\frac{\pi k a}{l} t\right) + \right.$$

$$\left. + D_2 \cos\left(\frac{\pi k a}{l} t\right)\right]$$

Используя нач. условия

$$\begin{cases} u(x,0) = A \sin\frac{\pi x}{l} = \sum\limits_{k=1}^{\infty} \sqrt{\frac{2}{l}} \sin\frac{\pi k}{l} x \cdot D_2 \\ u_t(x,0) = 0 = \sum\limits_{k=1}^{\infty} \sqrt{\frac{2}{l}} \sin\frac{\pi k}{l} x \cdot \left(D_1 \cos\frac{\pi k a}{l} t \cdot \frac{\pi k a}{l} - \right. \end{cases}$$

$$\left. - D_2 \sin\frac{\pi k a t}{l} \cdot \frac{l}{\pi k}\,\frac{\pi k a}{l}\right) = \sum_{k=1}^{\infty} \sqrt{\frac{2}{l}} \sin\frac{\pi k}{l} x\, D_1 \frac{\pi k a}{l} = 0$$

$D_1 = 0$

$$D_2 \sum_{k=1}^{\infty} \sqrt{\frac{2}{l}} \sin\frac{\pi k}{l}x = A \sin\frac{\pi x}{l} = Const$$

$$D_{2k} = \int_0^l A \sin\frac{\pi x}{l} \cdot \sqrt{\frac{2}{l}} \sin\frac{\pi k}{l} x\, dx = A\sqrt{\frac{2}{l}} \int_0^l \sin\frac{\pi x}{l} \sin\frac{\pi k}{l} x\, dx$$

$$= A\sqrt{\frac{2}{l}}\,\frac{1}{2}\int_0^l \left(\cos\left(\frac{\pi}{l} - \frac{\pi k}{l}\right)x - \cos\left(\frac{\pi}{l} + \frac{\pi k}{l}\right)x\right) dx =$$

$$= A \cdot \sqrt{\frac{2}{l}} \cdot \frac{1}{2}\left(\int_0^l \cos\frac{\pi(1-k)}{l}x\, dx - \int_0^l \cos\frac{\pi(1+k)}{l} x\, dx\right) =$$

$$= \sqrt{\frac{l}{2}}\, A.$$

$$u(x,t) = \sqrt{\frac{2}{l}} \sin\frac{\pi x}{l} \cdot \sqrt{\frac{l}{2}}\, A \cos\frac{\pi a}{l} t = A \sin\frac{\pi x}{l} \cos\frac{\pi a t}{l}$$

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н.у. $\begin{cases} u(x,0) = Ax, \ A = const \\ u_t(x,0) = 0 \end{cases}$ г.у. $\begin{cases} u(0,t) = 0 \\ u_x(l,t) = 0 \end{cases}$

№ 103 Найти продольные колебания стержня, один конец которого $(x=0)$ закреплен жестко, а другой $(x=l)$ - свободен при н.у. $u(x,0) = kx, \ u_t(x,0) = 0 \quad 0 \le x \le l$

н.у. $\begin{cases} u(x,0) = kx \\ u_t(x,0) = 0 \end{cases}$ г.у. $\begin{cases} u(0,t) = 0 \\ u_x(l,t) = 0 \end{cases}$

$$u_{tt} = a^2 u_{xx}$$

$$u(x,t) = X(x)T(t)$$

$$\frac{X''(x)}{X(x)} = \frac{1}{a^2}\frac{T''(t)}{T(t)} = -\lambda$$

$$\begin{cases} X''(x) + \lambda X(x) = 0 & X(x) \ne 0 \\ T''(t) + a^2\lambda T(t) = 0 & T(t) \ne 0 \end{cases}$$

Г. усл. дают

$$u(0,t) = X(0)T(t) = 0$$
$$u_x(l,t) = X'(l)T(t) = 0$$

$$\begin{cases} X(0) = 0 \\ X'(l) = 0 \end{cases}$$

I. $X''(x) + \lambda X(x) = 0, \quad \lambda > 0$

$$X(x) = A_1 \cos\sqrt{\lambda}x + A_2 \sin\sqrt{\lambda}x$$

$$X(0) = A_1 = 0.$$

$$X'(l) = -A_1 \sin\sqrt{\lambda}l + A_2 \cos\sqrt{\lambda}l = 0.$$

$$A_2 \cos\sqrt{\lambda}l = 0.$$

$A_2 \neq 0$

$\cos\sqrt{\lambda}\,l = 0$

$\sqrt{\lambda}\,l = \pi k + \dfrac{\pi}{2}$

$\sqrt{\lambda} = \dfrac{\pi k}{l} - \dfrac{\pi}{2l}$

$\lambda = \left(\dfrac{2\pi k + \pi}{2l}\right)^2 = \left(\dfrac{\pi(2k+1)}{2l}\right)^2$

$X_k(x) = A_2 \cos\dfrac{\pi(2k-1)}{2l}x$

Найдем A_2 из усл. норм/ки

$\|x\|^2 = 1 = \int\limits_0^l A_2^2 \cos^2\dfrac{\pi(2k+1)}{2l}x = A_2^2 \cdot \dfrac{1}{2}\Big(\int\limits_0^l \cos\dfrac{\pi(2k+1)}{l}x\,dx$

$-\int\limits_0^l dx\Big) = -A_2^2\dfrac{l}{2} + A_2^2 \cdot \dfrac{l}{2\pi(2k-1)} \sin\dfrac{\pi(2k+1)}{l}x\Big|_0^l =$

$= -A_2^2\dfrac{l}{2} + \dfrac{A_2^2 l}{2\pi(2k+1)} \sin\pi(2k+1) = -\dfrac{A_2^2}{2}l + \dfrac{A_2^2 l}{2\pi(2k-1)}\sin 2\pi n$

$= -\dfrac{A_2^2 l}{2}$ $\quad A_2 = \sqrt{\dfrac{l}{2}}\sqrt{\dfrac{2}{l}}$

II. $T''(t) + a^2\lambda T(t) = 0$

$T(t) = B_1\cos\dfrac{\pi(2k+1)}{2l}at + B_2\sin\dfrac{\pi(2k+1)}{2l}at$

$u(x,t) = \sum\limits_{k=1}^{\infty} X_k(x)T_k(t) = \sum\limits_{k=1}^{\infty} A_k \cos\dfrac{\pi(2k+1)}{2l}x\Big(B_1\cos\dfrac{\pi(2k+1)}{2l}at +$

$+ B_2\sin\dfrac{\pi(2k-1)}{2l}at\Big)$

$u(x,0) = \sqrt{\dfrac{2}{l}}\sum\limits_{k=0}^{\infty}\sin\dfrac{\pi(2k+1)}{2l}x \cdot B_{1k} = kx$

$u_t(x,0) = \sqrt{\dfrac{2}{l}}\sum\limits_{k=0}^{\infty}\sin\dfrac{\pi(2k+1)}{2l}x\cdot\dfrac{\pi(2k+1)}{2l}\cdot B_2 = 0$

$\underline{B_2 = 0}$

Пусть $\varphi(x) = Kx$.

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k \sin\frac{\pi(2k+1)}{2l}x, \text{ где}$$

$$\varphi_k = \frac{2}{l}\int_0^l \varphi(\xi)\sin\frac{\pi(2k+1)}{2l}\xi\, d\xi.$$

$$B_{1k} = \varphi_k$$

$\int f dg = fg - \int g df$

$$\frac{l}{2}B_{1k} = \frac{2}{l}\int_0^l K\xi \sin\underbrace{\frac{\pi(2k+1)}{2l}}_{=B}\xi\, d\xi = \begin{cases} \text{по частям} \\ f = \xi \quad dg = \sin B\xi \\ d\xi = dx \quad g = -\frac{\cos B\xi}{B} \end{cases}$$

$$= \frac{1}{B}\int_0^l \cos B\xi\, d\xi - \xi\frac{\cos B\xi}{B}\Bigg|_0^l =$$

$$= \frac{\sin B\xi}{B^2}\Bigg|_0^l - \xi\frac{\cos B\xi}{B}\Bigg|_0^l = \sin\frac{\pi(2k-1)}{2l}\xi\frac{4l^2}{\pi^2(2k-1)^2} -$$

$$- \xi\cos\frac{\pi(2k+1)}{2l}\xi\frac{2l}{\pi(2k-1)}\Bigg|_0^l =$$

$$= K\left(\sin\frac{\pi(2k+1)}{2}\cdot\frac{4l^2}{\pi^2(2k-1)^2} - l\cos\frac{\pi(2k+1)}{2}\cdot\frac{2l}{\pi(2k-1)} + \frac{l\cdot 2l}{\pi(2k+1)}\right.$$

$$= K\left((-1)^n\frac{4l^2}{\pi^2(2k+1)^2} - \frac{l\cdot 2l}{\pi(2k+1)} + \frac{l\cdot 2l}{\pi(2k+1)}\right) =$$

$$= \frac{K(-1)^k 4l^2}{\pi^2(2k+1)^2}.$$

$$\frac{l}{2}B_{1k} = \frac{K(-1)^k 4l^2}{\pi^2(2k+1)^2}.$$

$$B_{1k} = \frac{(-1)^k K\cdot 8\cdot l}{\pi^2(2k+1)^2}.$$

$$u(x,t)=\sum_{k=0}^{\infty}\sqrt{\frac{2}{l}}\cdot\frac{(-1)^k k\, 8l}{\pi^2(2k+1)^2}\cdot\sin\frac{\pi(2k+1)}{2l}x\cos\frac{\pi(2k+1)}{2l}at$$

$$=\frac{8kl}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\cdot\sin\frac{(2k+1)}{2l}\pi x\cdot\cos\pi\frac{(2k+1)}{2l}\pi a t$$

 # 104 Стержень с жестко-закрепленным концом $x=0$ растянутый под действием силы $F_0=\mathrm{const}$ ($x=l$) в сост. $t=0$ п/к-р

$t>0$

н.у. $\begin{cases} u(x,0)=kx, \text{ где } k=\dfrac{F_0}{ES} \\ u_t(x,0)=0 \end{cases}$ г.у. $\begin{cases} u(0,t)=0 \\ u_x(l,t)=0 \end{cases}$

$$u(x,t)=\frac{8F_0}{ES}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)}{2l}\pi x\cos\pi\frac{(2k+1)}{2l}\pi a t$$

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$u_{tt}=a^2u_{xx}$ $\quad 0<x<l,\ 0<t<\infty$

н.у. $u(x,0)=0$, $u_t(x,0)=0$ $\qquad$ г.у. $u(0,t)=0$, $u_x(l,t)=\dfrac{F_0}{ES}$

Неоднородные гранич. усл. => нужно сделать однородными.

$\underline{u(x,t)}=v(x,t)+\varphi(x)$

$u(0,t)=v(0,t)+\varphi(0)=0$

как в примере в т. $l=x$

φ: $u_x(l,t)=v_x(l,t)+\varphi(l)=v_x(l,t)+\dfrac{F_0}{ES}$

$v_x(l,t)=0\to$ одн.

по исходн. н.у.:

$u(x,0)=v(x,0)+\dfrac{F_0}{ES}$; $u(0,t)=v(0,t)=0.$

$v_{(x,0)}=-\dfrac{F_0}{ES}x$

III.22 Найти распределение температуры в стержне длины l с теплоизолированной боковой поверхностью, если температура ее концов поддерживается равной нулю, а начальная температура задана функцией, имеющей вид параболы, симметричной относительно центра стержня

$$U_t = a^2 U_{xx}$$

$$\text{г.у.} \begin{cases} u(0,t) = 0 \\ u(l,t) = 0 \end{cases} \qquad \text{н.у.} \ \{ \ u(x,0) = ax^2 + bx + c$$

$$U = X(x)\,T(t)$$

$$\begin{cases} X'' + \lambda X = 0 \\ T' + a^2 \lambda T = 0 \end{cases}$$

1) $$X = A_1 \cos(\sqrt{\lambda} x) + B_1 \sin(\sqrt{\lambda} x)$$

$$\begin{cases} X(0) = 0 \\ X(l) = 0 \end{cases} \quad \begin{matrix} A_1 = 0 \\ 0 = B_1 \sin(\sqrt{\lambda} l) \end{matrix}$$

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2$$

$$X_n = B_1 \sin \frac{\pi n}{l} x$$

$$\|X\| = \int_0^l B_1^2 \sin^2 \frac{\pi n}{l} x \, dx \; ; \quad \sqrt{\frac{2}{l}} = B_1$$

$$X_n = \sqrt{\frac{2}{l}} \sin \frac{\pi n}{l} x$$

2) $$T = C e^{-a^2 \lambda_n t} = C e^{-\frac{a^2 \pi^2 n^2}{l^2} t}$$

$$u(x,t) = \sum_n \sqrt{\frac{2}{l}} \sin \frac{\pi n}{l} x \cdot C e^{-\frac{a^2 \pi^2 n^2}{l^2} t}$$

$$A_n = \frac{2}{l}\int_0^l A\sin\frac{\pi x}{l}\sin\frac{\pi n x}{l}\,dx = \frac{l\sin\pi n}{\pi-\pi n^2}\cdot A\cdot\frac{2}{l} =$$

$$= \frac{2A\sin\pi n}{\pi-\pi n^2}$$

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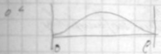

$0 \le x \le l$

Сила с момента $t = 0$.

$F(x,t) = Axt$ ($A = const$.

$F(x,t) = Axt$ ($A = const$)

$u_{tt} = a^2 u_{xx}$

Запишем граничные условия

Гран. условия.

$u(0,t)=0 \qquad u(x,0)=0$

$u(l,t)=0 \qquad u_t(x,0)=\Phi(x,t)$, где

$\Phi(x,t)=\dfrac{F(x,t)}{ES}$

$u=X(x)T(t)$

$$\begin{cases} X''+\lambda X=0 \\ T''+a^2\lambda T=0 \end{cases}$$

1) $X''+\lambda X=0$

$X=A_1\cos(\sqrt{\lambda}x)+B_1\sin(\sqrt{\lambda}x)$

Из ~~ее~~ гран условий

$$\begin{cases} u(0,t)=X(0)T(t)=0 \\ u(l,t)=X(l)T(t)=0 \end{cases} \qquad \begin{matrix} X(0)=0 \\ X(l)=0 \end{matrix}$$

$0=A_1=X(0)$

$X(l)=B_1\sin(\sqrt{\lambda}l)=0$

$$\lambda_n=\left(\frac{\pi n}{l}\right)^2$$

Тогда $X_n=B_1\sin\dfrac{\pi n}{l}x$

Найдем B_1. Для этого проведем нормировку

$$\|X\|^2=\int_0^l B_1^2\sin^2\left(\frac{\pi n}{l}x\right)dx=1.$$

$$\frac{B_1^2}{2}\int_0^l\left(1-\cos\frac{2\pi n}{l}x\right)dx=\frac{B_1^2 l}{2}-\frac{B_1^2\cdot l}{2\cdot 2\pi n}\sin\frac{2\pi n x}{l}\Big|_0^l.$$

$= \frac{B_1^2 l}{2} - \frac{B_1^2}{4\pi n} l \cdot \sin \pi n \cdot 2 = \frac{B_1^2 l}{2}$

$\frac{B_1^2 l}{2} = 1$; $B_1 = \sqrt{\frac{2}{l}}$

$X_n = \sqrt{\frac{2}{l}} \sin\left(\frac{\pi n}{l} x\right)$.

2). $T'' + a^2 \lambda T = 0$

$T = A_2 \cos\left(\frac{a\pi n}{l} t\right) + B_2 \sin\left(\frac{a \pi n}{l} t\right)$

Из н.у. $u(x,0) = X(x) T(0) = 0 \quad T(0) = 0$

$A_2 = 0$

$T = B_2 \sin\left(\frac{a \pi n}{l} t\right)$

$u = \sum_n X_n T_n = \sum_n \sqrt{\frac{2}{l}} \sin\left(\frac{\pi n}{l} x\right) \cdot B_2 \sin\left(\frac{a \pi n}{l} t\right)$

Найдем B_2

$u_t(x,0) = \sum_n \sqrt{\frac{2}{l}} \sin\left(\frac{\pi n}{l} x\right) \cdot B_2 \cdot \cos\left(\frac{a \pi n}{l} \cdot 0\right) \cdot \frac{\pi n a}{l}$

$B_2 \cdot \frac{\pi n a}{l} = \varphi$.

$\varphi = \frac{(X(x) \Phi(x,t))}{\|X\|^2} \quad (X(x)\, \Phi(x,t))$

$\frac{\pi n a}{l} B_2 = \int_0^l \sqrt{\frac{2}{l}} \sin\left(\frac{\pi n}{l} x\right) \cdot Axt \, dx$

1) IV.95
2) IV.113
4) 92
7) Задачи 2.1
10) I.52
11) II.4
13) II.127
16) II.128
17) 90
19) I.47
20) II.146
21) II.108
22) 6.47
25) V.28
30) V.29
34) V.49
38) I.2
40) I.3
42) I.5
43) I.15
45) I.16
46) I.17
47) I.21
48) II.83, метод разд перем.
51) 103
54) 104 и 128
55) III.22
56) II.146